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ENERGY METHODS FOR NONLINEAR HYPERBOLIC
VOLTERRA INTEGRODIFFERENTIAL EQUATIONS

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ABSTRACT

Technical summary rept.

We use energy methods to study global existence, boundedness, and asymptotic behavior as $t \to \infty$, of solutions of the two Cauchy problems (and related initial-boundary value problems)

$$\text{(HF)} \quad \begin{cases} u_t(t,x) = \int_0^t a(t-\tau)\sigma(u_x(\tau,x))_x \, d\tau + f(t,x) & (0 < t < \infty, \ x \in \mathbb{R}) \\[2mm] u(0,x) = u_0(x) & (x \in \mathbb{R}) , \end{cases}$$

$$\text{(VE)} \quad \begin{cases} u_{tt}(t,x) = \sigma(u_x(t,x))_x + \int_0^t a'(t-\tau)\sigma(u_x(\tau,x))_x \, d\tau + g(t,x) \\[2mm] \hspace{6cm} (0 < t < \infty, \ x \in \mathbb{R}) \\[2mm] u(0,x) = u_0(x), \ u_t(0,x) = u_1(x) & (x \in \mathbb{R}) \end{cases}$$

with suitably "small" data u_0, u_1, f, g; (HF) and (VE) are mathematical models for nonlinear one-dimensional heat flow in a material with "memory" and nonlinear one-dimensional viscoelastic motion, respectively. Here $a : [0,\infty) \to \mathbb{R}$, $\sigma : \mathbb{R} \to \mathbb{R}$, $f,g : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, $u_0, u_1 : \mathbb{R} \to \mathbb{R}$ are given, sufficiently smooth functions; the subscripts x or t denote partial derivatives. If $a(0) = 1$ formal differentiation with respect to t reduces (HF) to (VE) with $g(t,x) = f_t(t,x)$ and $u_1(x) = f(0,x)$. But, since (HF) and (VE) have different physical origins, the corresponding natural assumptions concerning $a(\cdot)$ are drastically different and, therefore, the two problems are studied separately.

A previous study of (HF) and (VE) rests on the concept of Riemann invariant and is restricted to one space dimension. The energy method is simpler in principle and yields more widely applicable results.

AMS (MOS) Subject Classifications: 45K05, 47H15, 47H10, 45M99, 35L60

Key Words: nonlinear Volterra partial integrodifferential equations (of hyperbolic type), energy method, smooth solutions, local and global existence and uniqueness, boundedness, asymptotic behavior, energy estimates, resolvent kernels, nonlinear heat flow, nonlinear viscoelastic motion, materials with memory

Work Unit # 1 (Applied Analysis)

SIGNIFICANCE AND EXPLANATION

This paper is devoted to a study of the two initial value problems (and some closely related boundary-initial value problems);

$$\text{(HF)}\begin{cases} u_t(t,x) = \int_0^t a(t-\tau)\sigma(u_x(\tau,x))_x d\tau + f(t,x) & (0 < t < \infty, \ x \in \mathbb{R}) \\ \\ u(0,x) = u_0(x) & (x \in \mathbb{R}), \end{cases}$$

$$\text{(VE)}\begin{cases} u_{tt}(t,x) = \sigma(u_x(t,x))_x + \int_0^t a'(t-\tau)\sigma(u_x(\tau,x))_x d\tau + g(t,x) \\ \hspace{8cm} (0 < t < \infty, \ x \in \mathbb{R}) \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x) & (x \in \mathbb{R}) \end{cases}$$

for suitably "small" data u_0, u_1, f, g. Here $a : [0,\infty) \to \mathbb{R}$, $\sigma : \mathbb{R} \to \mathbb{R}$, $f,g : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, $u_0, u_1 : \mathbb{R} \to \mathbb{R}$ are given real functions, and $a' = \frac{da}{dt}$. (HF) and (VE) are mathematical models for nonlinear heat flow in a material with memory and for nonlinear viscoelastic motion respectively. If $a(0) = 1$ formal differentiation of (HF) with respect to t reduces (HF) to (VE) with $g(t,x) = f_t(t,x)$, $u_1(x) = f(0,x)$. However, the different physical origins of the two problems, imply drastically different assumptions on the kernel $a(\cdot)$ for each, and therefore, the two problems are studied separately.

Problems (HF) and (VE) cannot in general be solved explicitly, even in the linear case $\sigma(r) = c^2 r$, c a constant; here the main interest is in more complicated nonlinear problems, e.g. $\sigma(r) = c^2 r + r^2$, since these provide more accurate mathematical models of physical situations. To suggest some of the difficulties consider the case $a(t) \equiv 1$ for which both (HF) and (VE) reduce to a nonlinear undamped wave equation (W) $u_{tt} = \sigma(u_x)_x + g$. It is known that (W), with the forcing term $g \equiv 0$, has the property that its solutions develop singularities in the first derivatives at some finite time t, no matter how smooth one takes the initial data; such solutions are called "shocks". Therefore, the initial value problem for (W) does not in general possess global, smooth solutions in time.

We use energy methods to establish the global existence, uniqueness, boundedness, and the decay as $t \to \infty$ of smooth solutions of (HF) and (VE), under physical reasonable assumptions concerning the "memory function" $a(\cdot)$, for smooth and suitably "small" data. One interpretation of our global existence results is that the presence of the integrals in (HF) and (VE) provides a damping mechanism which precludes the development of "shocks". The boundedness and decay results are relatively easy by products of the global existence results.

Due to the complexity of the equations under study any method of analysis will necessarily be quite technical. However, the energy method developed in this paper is simple, at least in principle, and it yields more general results for (HF) and (VE) than were obtained in a previous study by the method of Riemann invariants. For this reason our approach is not restricted to one space dimension, as we illustrate by outlining a two-dimensional version of (HF).

ENERGY METHODS FOR NONLINEAR HYPERBOLIC VOLTERRA INTEGRODIFFERENTIAL EQUATIONS

C. M. Dafermos [1],[2],[3] and J. A. Nohel [1],[4]

1. __Introduction.__ In this paper we use energy methods to study global existence, __bounded-ness__, and asymptotic behavior, as $t \to \infty$, of solutions of two initial value problems:

(HF)
$$\begin{cases} u_t(t,x) = \int_0^t a(t - \tau)\sigma(u_x(\tau,x))_x d\tau + f(t,x), & 0 < t < \infty, \; x \in R \\ u(0,x) = u_0(x), & x \in R , \end{cases}$$

(VE)
$$\begin{cases} u_{tt}(t,x) = \sigma(u_x(t,x))_x + \int_0^t a'(t - \tau)\sigma(u_x(\tau,x))_x d\tau + g(t,x), & 0 < t < \infty, \; x \in R \\ u(0,x) = u_0(x), \; u_t(0,x) = u_1(x), & x \in R \end{cases}$$

with suitably "small" data. Here $a : [0,\infty) \to R$, $\sigma : R \to R$, $f,g : [0,\infty) \times R \to R$, $u_0, u_1 : R \to R$ are given functions; subscripts x or t denote corresponding partial derivatives; a prime (') denotes the derivative of functions of a single variable.

Problem (HF) represents a mathematical model for heat flow in unbounded one-dimensional bodies of material with memory while (VE) is a model for the equation of motion of an unbounded one-dimensional nonlinear viscoelastic body. The corresponding initial-boundary value problems for bounded bodies have been studied by MacCamy [7], [8]; we also refer to [7], [8] for a sketch of the derivation of the equations from physical principles. Here we are dealing primarily with the initial value problem but in Section 6 we show that our methods apply equally well to certain initial-boundary value problems.
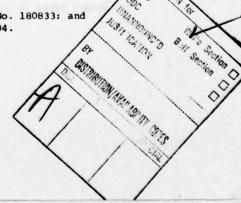
About $\sigma(\cdot)$ we make the assumptions

(σ) $\qquad\qquad\qquad\qquad\qquad\qquad \sigma \in C^3(R), \; \sigma(0) = 0, \; \sigma'(0) > 0$

the first for technical reasons and the other two on physical grounds (in the linear

versions of (HF) and (VE) $\sigma(u_x) = u_x$. Concerning the forcing terms we assume

(f) $\qquad\qquad f, f_t, f_x, f_{tt}, f_{tx}, f_{xx}, f_{ttt}, f_{ttx}, f_{txx} \in L^2([0,\infty); L^2(R))$,

(g) $\qquad\qquad g, g_t \in L^1([0,\infty); L^2(R)), \; g_x, g_{tt}, g_{tx} \in L^2([0,\infty); L^2(R))$

meaning that f, g and their (distributional) derivatives are endowed with some smooth-
ness and decay sufficiently rapidly at infinity. The initial datum $u_0(x)$ in both (HF)
and (VE) will be assumed to satisfy

(u_0) $\qquad\qquad\qquad\qquad u_{0x}, u_{0xx}, u_{0xxx} \in L^2(R)$

while $u_1(x)$ in (VE) will satisfy

(u_1) $\qquad\qquad\qquad\qquad u_1, u_{1x}, u_{1xx} \in L^2(R)$.

We shall postpone to Section 2 the precise assumptions on the kernels $a(t)$. For the
moment it suffices to know that $a(t)$ is C^3 smooth and, without loss of generality,
$a(0) = 1$.

Formal differentiation of (HF) with respect to t reduces this problem to (VE)
with $g(t,x) = f_t(t,x)$ and $u_1(x) = f(0,x)$. However, since (HF) and (VE) have different
physical origins, the corresponding natural assumptions on $a(\cdot)$ are drastically
different (see Section 2) and as a result the above problems have to be studied separately.

If $a(t) = 1$, $t \in [0,\infty)$, both (HF) and (VE) reduce to a nonlinear undamped wave
equation $u_{tt} = \sigma(u_x)_x + g$. For the latter (take $g \equiv 0$) it is known [4] that
the initial value problem does not generally have global smooth solutions, no matter
how smooth the initial data are. However, for the wave equation with "frictional"
damping, $u_{tt} + u_t = \sigma(u_x)_x$, Nishida [11] shows that when the initial data are "small",
the dissipation precludes the development of shocks, and as a result global smooth
solutions to the initial value problem exist. The proof rests heavily on the concept
of Riemann invariants and is strictly "one-space-dimensional". In the aforementioned
papers [7], [8], MacCamy shows that, under natural assumptions on the kernels $a(t)$,
the memory terms in (HF) and (VE) induce dissipative mechanisms that guarantee the

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existence of global smooth solutions when the initial data and the forcing term are "small". The proof combines certain energy integrals with Nishida's Riemann invariants argument and consequently is "one-space-dimensional". For corrections of certain misprints and oversights in [7], [8] see Section 2. The question of obtaining the existence and uniqueness of a suitable local solution for (HF) and (VE), to be continued with the aid of a priori estimates, is not discussed in [7], [8], but this gap can be filled by the method outlined in Nohel [12]. For the multi-space-dimensional nonlinear wave equation with frictional damping and "small" data Matsumura [9], [10] establishes the existence of global smooth solutions by a method that is based exclusively on "energy" estimates. (We are grateful to Professor Nishida for explaining this method to us.) The object of this paper is to study (HF) and (VE) by a similar approach. We are restricting our attention to the one-space-dimensional situation for clarity - the method seems to work in any number of space dimensions; a two-dimensional version of (HF) is discussed briefly in Section 7.

Our procedure can be outlined as follows: In Section 2 we reduce, similar to [7], [8], both (HF) and (VE) to the equivalent form

$$(1.1) \quad \begin{cases} u_{tt}(t,x) + \dfrac{\partial}{\partial t} \displaystyle\int_0^t k(t - \tau) u_t(\tau,x) d\tau = \sigma(u_x(t,x))_x + \phi(t,x), \quad 0 < t < \infty, \; x \in R \\ u(0,x) = u_0(x), \; u_t(0,x) = u_1(x), \; x \in R \end{cases}$$

where $k(t)$ is the resolvent kernel associated with $a'(t)$ (see Section 2) and $\phi(t,x)$, determined by $k(t)$ and $f(t,x)$ or $g(t,x)$, satisfies

$$(\phi) \qquad \qquad \phi, \phi_t, \phi_x, \phi_{tt}, \phi_{tx} \in L^2([0,\infty); L^2(R))$$

for problem (HF), and some additional conditions for (VE) (see conditions (ϕ_v), Section 2). In Section 3 we prove with the help of the Banach fixed point theorem an existence and uniqueness theorem of a local solution to (1.1) that applies both to (HF) and (VE). In Sections 4 and 5 we establish for (HF) and (VE), respectively, "energy" estimates that allow the extension of the local solution, constructed in Section 3, into a global solution. These estimates have the form

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(1.2) $\quad E(t) - E(0) \leq - \int\limits_{0}^{t} \int\limits_{-\infty}^{\infty} Q[u,u]dxd\tau + \int\limits_{0}^{t} \int\limits_{-\infty}^{\infty} P[u,u]dxd\tau + \int\limits_{0}^{t} \int\limits_{-\infty}^{\infty} \Pi[u,\phi]dxd\tau$

where $E(t)$ is an "energy" that controls the growth of the solution; $Q[u,u]$, the dissipation term induced by the memory term, is a positive definite quadratic form in a set of derivatives of $u(t,x)$; $P[u,u]$, the remainder term due to the nonlinearity of the problem, is a quadratic form in the same derivatives as $Q[u,u]$ and with coefficients that are small whenever the "energy" E is small; finally, $\Pi[u,\phi]$ is a bilinear form in the set of derivatives of $u(t,x)$ involved in $Q[u,u]$ and in $\phi(t,x)$ and some of its derivatives. The idea now is that for as long as $E(t)$ is small, $P[u,u]$ is dominated by $-Q[u,u]$. Moreover, the Cauchy-Schwarz inequality allows us to dominate the u-part in $\Pi[u,\phi]$ by $-Q[u,u]$. Then, if $E(0)$ and ϕ are "small", (1.2) shows that $E(t)$ remains small and the cycle closes.

Finally, we note that both problems (HF) and (VE) are of the abstract form

(A) $\quad \begin{cases} u''(t) + a(0)Au(t) + \int\limits_{0}^{t} a'(t - \tau)Au(\tau)d\tau = F(t), & 0 < t < \infty \\[1em] u(0) = u_0, \ u'(0) = u_1 \end{cases}$

where A is a nonlinear maximal monotone operator in a Hilbert space H, $F(t)$ takes values in H while $u(t)$ takes values in a reflexive Banach space W dense in H. The global existence problem for (A) was extensively studied by S. O. Londen [5], [6] for a class of kernels $a(\cdot)$ which are positive, smooth, decreasing, convex on $[0,\infty)$ and which satisfy the crucial condition $a'(0+) = -\infty$. Unfortunately, this last condition is not satisfied by most memory functionals arising in heat flow theory or in visco-elasticity.

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2. __Properties of Resolvent Kernels and Transformation of Problems (HF) and (VE)__. We

first show how (HF) and (VE) can be brought to the form (1.1). We define the resolvent

kernel $k(\cdot)$ associated with $a'(\cdot)$ via the equation

(k) $k(t) + (a'*k)(t) = -a'(t), \quad 0 \leq t < \infty$,

where, throughout, the * will denote the convolution, i.e.,

$$(a'*k)(t) = \int_0^t a'(t - \tau)k(\tau)d\tau .$$

By standard Volterra equations theory, if $a \in C^3$ smooth, $k(\cdot)$ is uniquely defined

and is C^2 smooth on $[0,\infty)$ (see, e.g., Bellman and Cooke [2, Thm. 7.4]). Moreover,

for any $\varphi \in L^1_{loc}(0,\infty)$, the unique solution of the Volterra equation

(2.1) $y(t) + (a'*y)(t) = \varphi(t), \quad 0 \leq t < \infty$,

is given by

(2.2) $y(t) = \varphi(t) + (k*\varphi)(t), \quad 0 \leq t < \infty$.

We now visualize the equation in (VE) as a Volterra equation of the form (2.1)

with $y = \sigma(u_x)_x$ so that (2.2) yields

(2.3) $u_{tt}(t,x) + (k*u_{tt})(t,x) = \sigma(u_x(t,x))_x + g(t,x) + (k*g)(t,x) .$

An integration by parts with respect to t in the convolution term on the left-hand

side of (2.3) shows that (VE) is equivalent to (1.1) with

(2.4)$_V$ $\phi(t,x) = g(t,x) + (k*g)(t,x) + k(t)u_1(x) .$

Problem (HF) is treated in a similar way. First we differentiate the equation in

(HF) with respect to t thus bringing it to the form

$$u_{tt}(t,x) = \sigma(u_x)_x(t,x) + (a'*\sigma(u_x)_x)(t,x) + f_t(t,x) ;$$

then we use (2.2) to get

$$u_{tt}(t,x) + (k*u_{tt})(t,x) = \sigma(u_x(t,x))_x + f_t(t,x) + (k*f_t)(t,x)$$

and, finally, we integrate by parts with respect to t in the convolution terms to

arrive at (1.1) with

(2.4)$_H$ $\phi(t,x) = f_t(t,x) + k(0)f(t,x) + (k'*f)(t,x), \quad 0 \leq t < \infty, \ x \in R$

(2.5)$_H$ $u_1(x) = f(0,x), \quad x \in R .$

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We note that, on account of assumptions (f), $u_1(x)$, given by $(2.5)_H$, satisfies assumption (u_1).

We now state for each case, separately, the assumptions on the kernels $a(\cdot)$ and we derive the induced properties of the associated resolvent kernels.

I. Heat Flow Equation

We assume

(a_H)
$$\begin{cases}
\text{(i)} & a(t) \in C^3[0,\infty), \; a(t),a'(t),a''(t),a'''(t) \quad \text{are bounded on} \quad [0,\infty) \\
\text{(ii)} & a(0) = 1, \; a'(0) < 0 \\
\text{(iii)} & t^j a^{(m)}(t) \in L^1(0,\infty), \quad j = 0,1,2,3, \quad m = 0,1,2,3 \\
\text{(iv)} & \text{Re} \; \hat{a}(i\eta) > 0, \; \eta \in R, \quad \text{where} \quad \hat{a}(s) = \int_0^\infty e^{-st} a(t)dt \; .
\end{cases}$$

We note that there is no loss of generality in assuming $a(0) = 1$, provided $a(0) > 0$. If $a(0) \neq 1$, equation (k) is modified to:

$$a(0)k(t) + (a'*k)(t) = -\frac{a'(t)}{a(0)} \; ;$$

(2.2) and all subsequent equations involving k are not affected by this change in an essential way. The following proposition summarizes properties of the resolvent kernel $k(t)$ associated with $a'(t)$.

Lemma 2.1. Assume that (a_H) are satisfied and let $k(t)$ be the resolvent kernel associated with $a'(t)$. Then

(i) $k(t) \in C^2[0,\infty)$; $k(t),k'(t),k''(t)$ are bounded on $[0,\infty)$.

(ii) $k(t) = k_\infty + K(t)$; $k_\infty = \dfrac{1}{\hat{a}(0)} > 0$; $K^{(m)}(t) \in L^1(0,\infty)$, $m = 0,1,2$.

(iii) For any $T > 0$ there is a number $\alpha > 0$ such that

$(2.6)_H$
$$\int_0^T v(t) \frac{d}{dt} (k*v)(t)dt \geq \alpha \int_0^T v^2(t)dt$$

for every $v(t) \in L^2(0,T)$.

Assertion (iii) of Lemma 2.1 is the manifestation of the dissipative character of the memory term and will play a central role in Section 4. For the (harmonic analysis) proof of Lemma 2.1 we refer to [7, Lemma 3.1]. We note that our assumptions

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(a_H) are weaker than the corresponding assumptions (a_2) in [7], the reason being that our method does not require that moments of K and K' be in $L^1(0,\infty)$ (see [7, Lemma 3.1, assertion (ii)]). The reader of the proof of Lemma 3.1 in [7] should be aware of the following misprints: Formula (3.5) should read

$$\hat{k}(s) = - \frac{\hat{a}(s) - \hat{a}(0)}{s\hat{a}(s)\hat{a}(0)} - \frac{1}{\hat{a}(0)} \ .$$

Moreover, the assumption in [7] that $a(t) \in C^2[0,\infty)$ is insufficient since it only yields $o\left(\frac{1}{s}\right)$ for the error term in [7, eq. (3.8)] rather than $O\left(\frac{1}{s^2}\right)$, as it is needed. It is for this reason that we are assuming here $a(t) \in C^3[0,\infty)$. The proof of (iii) can be accomplished more simply by the technique of [13, Theorem 1] than that of [7]; the same comment applies to the proof of (2.11) below.

We note that assumption (f) together with $k''(t) \in L^\infty(0,\infty)$ and $k'(t), k''(t) \in L^1(0,\infty)$ (Lemma 2.1) yield that $\phi(t,x)$, as defined by $(2.4)_H$, satisfies condition (ϕ) recorded in Section 1.

II. Viscoelasticity Equation

We make the following assumptions concerning the kernel $a(t)$ in (VE):

(a_V)
$\begin{cases} \text{(i)} \quad a(t) \in C^3[0,\infty), \ a(t), a'(t), a''(t), a'''(t) \quad \text{are bounded on } [0,\infty) \\ \text{(ii)} \quad a(t) = a_\infty + A(t), \ a_\infty > 0, \ a(0) = 1 \\ \text{(iii)} \quad (-1)^m A^{(m)}(t) \geq 0, \ 0 \leq t < \infty, \ m = 0,1,2; \ A'(t) \neq 0 \\ \text{(iv)} \quad t^j A^{(m)}(t) \in L^1(0,\infty), \quad j = 0,1,2,3, \quad m = 0,1,2,3 \ . \end{cases}$

We note that (a_V)(iii) implies (a_H)(iv) by a standard result [13, Cor. 2.2]. The difference between (a_H) and (a_V) that has a major effect on the properties of the corresponding resolvent kernels is that in the former $a(\infty) = 0$ while in the latter $a(\infty) = a_\infty > 0$. In the place of Lemma 2.1 we now have

Lemma 2.2. *Assume that* (a_V) *are satisfied and let* $k(t)$ *be the resolvent kernel associated with* $a'(t)$. *Then*

(i) $k(t) \in C^2[0,\infty)$; $k(t), k'(t), k''(t)$ *are bounded on* $[0,\infty)$.

(ii) $k^{(m)}(t) \in L^1(0,\infty), \quad m = 0,1,2$.

(iii) *For any* $T > 0$ *and every* $v(t) \in L^2(0,T)$,

$(2.6)_V$
$$\int_0^T v(t) \frac{d}{dt} (k*v)(t) dt \geq 0 \ .$$

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For a proof of Lemma 2.2 see [8, Lemma 3.1].

We note that assumption (g) together with $k''(t) \in L^\infty(0,\infty)$ and

$k(t),k'(t),k''(t) \in L^1(0,\infty)$ imply that $\phi(t,x)$, defined by $(2.4)_V$, satisfies

(ϕ_V) $\qquad\qquad \phi,\phi_t \in L^1([0,\infty); L^2(R))$, $\phi_x,\phi_{tt},\phi_{tx} \in L^2([0,\infty); L^2(R))$,

namely conditions that imply (ϕ) of Section 1.

In contrast to $(2.6)_H$, $(2.6)_V$ only indicates a weak dissipative mechanism. Indeed, the dissipative mechanism for the viscoelasticity equation is quite subtle and it will reveal itself through a device of MacCamy [8] that involves still another form of (VE). We define a function $r : (0,\infty) \to \mathbb{R}$ by

(r) $$r(t) = \beta + k(t) + \beta \int_0^t k(\tau)d\tau$$

where $\beta > 0$ is a constant to be specified below and $k(t)$ is the resolvent kernel associated with $a'(t)$. It is easily verified that the solution $y(t)$ of the Volterra equation (2.1) satisfies

(2.7) $$y(t) + \beta \int_0^t y(\tau)d\tau = \varphi(t) + (r*\varphi)(t), \quad 0 \le t < \infty .$$

Since $y = \sigma(u_x)_x$ in (VE) satisfies an equation of the form (2.1) we obtain from (2.7)

(2.8) $\quad u_{tt}(t,x) + (r*u_{tt})(t,x) = \sigma(u_x(t,x))_x + \beta \int_0^t \sigma(u_x(\tau,x))_x d\tau + g(t,x) + (r*g)(t,x)$.

Thus (VE) is equivalent to the problem

(2.9) $$\begin{cases} u_{tt}(t,x) + (r*u_{tt})(t,x) = \sigma(u_x(t,x))_x + \beta \int_0^t \sigma(u_x(\tau,x))_x d\tau + \Psi(t,x), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le t < \infty, \quad x \in R \\ u(0,x) = u_0(x), \; u_t(0,x) = u_1(x), \quad x \in R \end{cases}$$

where

(2.10) $$\Psi(t,x) = g(t,x) + (r*g)(t,x) .$$

The justification for considering the complicated variant (2.9) of (VE) is provided by the following proposition:

<u>Lemma 2.3.</u> <u>Assume that assumptions (a_V) are satisfied and let $r(t)$ be defined by</u>

(r). <u>Then</u>

(i) $r(t) \in C^2[0,\infty)$, $r(t),r'(t),r''(t)$ <u>are bounded on</u> $[0,\infty)$.

(ii) $r(t) = r_\infty + R(t)$; $r_\infty = \beta/a_\infty$; $R^{(m)}(t) \in L^1(0,\infty)$, $m = 0,1,2$.

(iii) <u>For any</u> $T > 0$ <u>there are constants</u> $\gamma,q > 0$, <u>with</u> $\beta q < 1$, <u>such that</u>

(2.11) $$q \int_0^T v(t) \frac{d}{dt} (r*v)(t)dt - \int_0^T v(t)(R*v)(t)dt \geq (1 + \gamma) \int_0^T v^2(t)dt$$

<u>for every</u> $v(t) \in L^2(0,T)$.

It is (2.11) that reveals the dissipative mechanism induced by the memory term in the viscoelasticity equations. This estimate will play a crucial role in Section 5.

For the proof of Lemma 2.3 we refer to [8, Lemma 3.2]. For the benefit of the reader we record here the following corrections in the proof of [8, Proposition 4.1]: Equation (4.16) should read

$$-q\eta \operatorname{Im} \hat{r}(i\eta) - \operatorname{Re} \hat{r}(i\eta) = \Gamma(q,\beta,\eta) + 1, \quad \eta \in R ;$$

equation (4.26) should read

$$\frac{1}{q} - \beta\gamma = \frac{m(\eta)q^2 + n(\eta)}{q(n(\eta) - q)}$$

and the sentence following this equation should read: "Given $\bar{q}$, $\bar{Q}$, $0 < \bar{q} < \bar{Q} < \infty$, there exists an $\bar{\epsilon} > 0$, $\bar{\epsilon} = \bar{\epsilon}(\bar{q},\bar{Q})$, such that for every $\bar{q} \leq q \leq \bar{Q}$ (4.27) holds". Note also that $\frac{1}{q_k} - \beta\gamma(q_k,\eta_k)$ tends to $-\dot{a}(0)$ (and not to $-\dot{a}(0)/q^2$). In the concluding argument of the proposition one needs to choose $0 < \epsilon < \min(\bar{\epsilon}(\bar{q},\bar{Q}), -2\dot{a}(0))$ in order to carry through the proof, since $\gamma(q,\beta,\eta) \to q\beta - 1 - \dot{a}(0)q$ and $0 < \frac{1}{q} - \beta < \frac{\epsilon}{2}$, by the choice of q and β.

We close this section by noting that on account of (2.10), assumption (g) and $r'(t),r''(t) \in L^1(0,\infty)$ (Lemma 2.3), the forcing term $\Psi(t,x)$ in (2.9) satisfies

(Ψ_V) $\qquad\qquad\qquad \Psi_t \in L^1([0,\infty); L^2(R))$, $\Psi_{tt},\Psi_{tx} \in L^2([0,\infty); L^2(R))$.

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3. Local Existence Theorem. We discuss here existence and uniqueness of local solutions
to problem (1.1), applicable to both (HF) and (VE). Throughout this section we will
be assuming that the initial data satisfy conditions (u_0), (u_1) and the forcing
term satisfies (Φ). As regards the kernel $k(t)$, we only require that
$k'(t), k''(t) \in C[0,\infty) \cap L^1(0,\infty)$ so that our analysis covers both the (HF) and the (VE)
cases. On the other hand, since the local solution will not be necessarily "small",
for the well-posedness of the problem we have to replace (σ) by the stricter assumption

$(\sigma)^*$ $\qquad\qquad\qquad \sigma \in C^3(R); \; \sigma(0) = 0; \; \sigma'(w) \geq p_0 > 0, \; w \in (-\infty,\infty)$.

This assumption will be dropped in Sections 4 and 5 where we will limit ourselves to
"small" solutions.

The main result of this section is

Theorem 3.1. Let the assumptions (σ^*), (Φ), (u_0), (u_1). and $k', k'' \in C[0,\infty) \cap L^1(0,\infty)$
be satisfied. There is a unique solution $u(t,x) \in C^2([0,T_0) \times R)$ of (1.1) defined
on a maximal interval $[0,T_0)$, $T_0 \leq \infty$, such that, for $T \in [0,T_0)$,

(3.1) $\qquad u_t, u_x, u_{tt}, u_{tx}, u_{xx}, u_{ttt}, u_{ttx}, u_{txx}, u_{xxx} \in L^\infty([0,T]; L^2(R))$.

Furthermore, if $T_0 < \infty$, then

(3.2) $\displaystyle\int_{-\infty}^{\infty} [u_t^2(t,x) + u_x^2(t,x) + u_{tt}^2(t,x) + u_{tx}^2(t,x) + u_{xx}^2(t,x) + u_{ttt}^2(t,x) + u_{ttx}^2(t,x)$

$\qquad\qquad + u_{txx}^2(t,x) + u_{xxx}^2(t,x)]dx \to \infty, \quad t \to T_0$.

The proof of the above proposition will be based on an application of the Banach
fixed point theorem. We begin with some preparation.

For positive M and T we let $X(M,T)$ denote the set of functions
$u(t,x) \in C^2([0,T] \times R)$, with initial conditions $u(0,x) = u_0(x)$, $u_t(0,x) = u_1(x)$,
which satisfy (3.1), and

(3.3) $\displaystyle\sup_{[0,T]} \int_{-\infty}^{\infty} [u_t^2 + u_x^2 + u_{tt}^2 + u_{tx}^2 + u_{xx}^2 + u_{ttt}^2 + u_{ttx}^2 + u_{txx}^2 + u_{xxx}^2]dx \leq M^2$.

Note that $X(M,T)$ is nonempty if M is sufficiently large. Also observe that for

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$u(t,x) \in X(M,T)$ (3.3) easily yields

(3.4) $\displaystyle\sup_{[0,T]\times R} \{|u_t(t,x)|, |u_x(t,x)|, |u_{tt}(t,x)|, |u_{tx}(t,x)|, |u_{xx}(t,x)|\} \leq M$

(e.g. $u_t^2(t,y) = \displaystyle\int_{-\infty}^{y} [u_t^2(t,x)]_x dx \leq 2 \int_{-\infty}^{\infty} |u_t(t,x) u_{tx}(t,x)| dx \leq \int_{-\infty}^{\infty} [u_t^2(t,x) + u_{tx}^2(t,x)] dx \leq M^2,$

and similarly for the others).

We now construct a map $S : X(M,T) \to C^2([0,T] \times R)$ which carries $v(t,x) \in X(M,T)$
into the solution $u(t,x)$ of the linear initial value problem

(3.5)
$$\begin{cases} u_{tt}(t,x) + k(0)u_t(t,x) = \sigma'(v_x(t,x))u_{xx}(t,x) + \Phi(t,x) - (k'*v_t)(t,x), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leq t < \infty, \ x \in R \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x), \ x \in R . \end{cases}$$

Our goal is to show that S has a unique fixed point in $X(M,T)$ since the desired
solution of (1.1) is such a fixed point (note that
$\frac{\partial}{\partial t} (k*u_t)(t,x) = k(0)u_t(t,x) + (k'*u_t)(t,x)).$

<u>Lemma 3.1.</u> If M is sufficiently large and T sufficiently small, then S maps
$X(M,T)$ to itself.

<u>Proof.</u> We have to show that the solution of (3.5) satisfies (3.1), (3.3), provided
that M is large and T is small. To this end we establish below a number of a priori
"energy" estimates for solutions of (3.5).

Let us assume, temporarily, that $\sigma(\cdot)$, $k(t)$, $u_0(x)$, $u_1(x)$, $\Phi(t,x)$ and $v(t,x)$
are C^∞ smooth on the corresponding domains of definition and that $u_0(\cdot)$, $u_1(\cdot)$, $\Phi(t,\cdot)$
and $v(t,\cdot)$ are compactly supported on R. Then the solution $u(t,x)$ of (3.5) will
be C^∞ smooth on $[0,\infty) \times R$ and $u(t,\cdot)$ will have compact support in R for $t \geq 0$.

Multiplying the equation in (3.5) by $u_t(t,x)$ and integrating over $[0,s] \times R$,
$0 \leq s \leq T$, we obtain, after an integration by parts with respect to x and other
straightforward calculations (for simplicity we omit the arguments of functions whenever
no confusion arises from doing so in this and subsequent calculations),

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(3.6) $\frac{1}{2} \int_{-\infty}^{\infty} u_t^2(s,x)dx + \frac{1}{2} \int_{-\infty}^{\infty} \sigma'(v_x(s,x))u_x^2(s,x)dx - \frac{1}{2} \int_{-\infty}^{\infty} u_t^2(0,x)dx$

$$- \frac{1}{2} \int_{-\infty}^{\infty} \sigma'(v_x(0,x))u_x^2(0,x)dx = - \int_0^s \int_{-\infty}^{\infty} k(0)u_t^2 dxdt$$

$$+ \int_0^s \int_{-\infty}^{\infty} \frac{1}{2} \sigma''(v_x)v_{tx}u_x^2 dxdt - \int_0^s \int_{-\infty}^{\infty} \sigma''(v_x)v_{xx}u_t u_x dxdt$$

$$+ \int_0^s \int_{-\infty}^{\infty} \Phi u_t dxdt - \int_0^s \int_{-\infty}^{\infty} (k'*v_t)u_t dxdt .$$

To aid the reader we indicate a calculation contained in (3.6); similar calculations are involved in (3.7)-(3.10) as well as in Sections 4, 5, 6, 7. Integrating by parts with respect to x one has

$$\int_0^s \int_{-\infty}^{\infty} \sigma'(v_x)u_{xx}u_t dxdt = - \int_0^s \int_{-\infty}^{\infty} \sigma'(v_x)u_{tx}u_x dxdt - \int_0^s \int_{-\infty}^{\infty} \sigma''(v_x)v_{xx}u_t u_x dxdt .$$

Observing that $u_x u_{tx} = \frac{1}{2} \frac{\partial}{\partial t} u_x^2$ and integrating the first integral on the right side by parts with respect to t yields

$$\int_0^s \int_{-\infty}^{\infty} \sigma'(v_x)u_{xx}u_t dxdt = - \int_0^s \frac{\partial}{\partial t} \{\int_{-\infty}^{\infty} \frac{1}{2} \sigma'(v_x)u_x^2 dx\}dt$$

$$+ \frac{1}{2} \int_0^s \int_{-\infty}^{\infty} \sigma''(v_x)v_{tx}u_x^2 dxdt - \int_0^s \int_{-\infty}^{\infty} \sigma''(v_x)v_{xx}u_t u_x dxdt ,$$

which, together with the other terms resulting from multiplying (3.5) by u_t and integrating, easily gives (3.6).

We next differentiate the equation in (3.5) with respect to t obtaining

$$u_{ttt} + k(0)u_{tt} = \sigma'(v_x)u_{xxt} + \sigma''(v_x)v_{tx}u_{xx} + \Phi_t - k'(0)v_t - (k''*v_t)(t,x) .$$

We multiply this equation by $u_{tt}(t,x)$, and following the procedure used in the derivation of (3.6) we obtain

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(3.7) $\quad \frac{1}{2} \int_{-\infty}^{\infty} u_{tt}^2(s,x)dx + \frac{1}{2} \int_{-\infty}^{\infty} \sigma'(v_x(s,x))u_{tx}^2(s,x)dx - \frac{1}{2} \int_{-\infty}^{\infty} u_{tt}^2(0,x)dx$

$\quad - \frac{1}{2} \int_{-\infty}^{\infty} \sigma'(v_x(0,x))u_{tx}^2(0,x)dx = - \int_0^s \int_{-\infty}^{\infty} k(0)u_{tt}^2 dxdt + \int_0^s \int_{-\infty}^{\infty} \frac{1}{2}\sigma''(v_x)v_{tx}u_{tx}^2 dxdt$

$\quad - \int_0^s \int_{-\infty}^{\infty} \sigma''(v_x)v_{xx}u_{tt}u_{tx}dxdt + \int_0^s \int_{-\infty}^{\infty} \sigma''(v_x)v_{tx}u_{tt}u_{xx}dxdt + \int_0^s \int_{-\infty}^{\infty} \phi_t u_{tt}dxdt$

$\quad - \int_0^s \int_{-\infty}^{\infty} k'(0)v_t u_{tt}dxdt - \int_0^s \int_{-\infty}^{\infty} (k''*v_t)u_{tt}dxdt \; .$

We now differentiate the equation in (3.5) with respect to x, we multiply by $u_{tx}(t,x)$ and we follow the above standard procedure to get

(3.8) $\quad \frac{1}{2} \int_{-\infty}^{\infty} u_{tx}^2(s,x)dx + \frac{1}{2} \int_{-\infty}^{\infty} \sigma'(v_x(s,x))u_{xx}^2(s,x)dx - \frac{1}{2} \int_{-\infty}^{\infty} u_{tx}^2(0,x)dx$

$\quad - \frac{1}{2} \int_{-\infty}^{\infty} \sigma'(v_x(0,x))u_{xx}^2(0,x)dx = - \int_0^s \int_{-\infty}^{\infty} k(0)u_{tx}^2 dxdt$

$\quad + \int_0^s \int_{-\infty}^{\infty} \frac{1}{2}\sigma''(v_x)v_{tx}u_{xx}^2 dxdt + \int_0^s \int_{-\infty}^{\infty} \phi_x u_{tx}dxdt - \int_0^s \int_{-\infty}^{\infty} (k'*v_{tx})u_{tx}dxdt \; .$

The next estimate is obtained by taking the second derivative of the equation in (3.5) with respect to t and then multiplying by $u_{ttt}(t,x)$. In the present case note from the equation preceding (3.7) that $\frac{\partial}{\partial t}(k''*v_t)(t,x) = k''(t)v_t(0,x) + (k''*v_{tt})(t,x)$. The result of the calculation is

(3.9) $\quad \frac{1}{2} \int_{-\infty}^{\infty} u_{ttt}^2(s,x)dx + \frac{1}{2} \int_{-\infty}^{\infty} \sigma'(v_x(s,x))u_{ttx}^2(s,x)dx - \frac{1}{2} \int_{-\infty}^{\infty} u_{ttt}^2(0,x)dx$

$\quad - \frac{1}{2} \int_{-\infty}^{\infty} \sigma'(v_x(0,x))u_{ttx}^2(0,x)dx = - \int_0^s \int_{-\infty}^{\infty} k(0)u_{ttt}^2 dxdt$

$\quad + \int_0^s \int_{-\infty}^{\infty} \frac{1}{2}\sigma''(v_x)v_{tx}u_{ttx}^2 dxdt - \int_0^s \int_{-\infty}^{\infty} \sigma''(v_x)v_{xx}u_{ttt}u_{ttx}dxdt$

$\quad + \int_0^s \int_{-\infty}^{\infty} 2\sigma''(v_x)v_{tx}u_{ttt}u_{txx}dxdt + \int_0^s \int_{-\infty}^{\infty} \sigma''(v_x)v_{ttx}u_{xx}u_{ttt}dxdt$

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$$+ \int_0^s \int_{-\infty}^{\infty} \sigma'''(v_x)v_{tx}^2 u_{xx}u_{ttt} + \int_0^s \int_{-\infty}^{\infty} \phi_{tt}u_{ttt}dxdt - \int_0^s \int_{-\infty}^{\infty} k'(0)v_{tt}u_{ttt}dxdt$$

$$- \int_0^s \int_{-\infty}^{\infty} k''(t)v_t(0,x)u_{ttt}dxdt - \int_0^s \int_{-\infty}^{\infty} (k''*v_{tt})u_{ttt}dxdt .$$

The final estimate is obtained by taking the second derivative of the equation in (3.5) with respect to t and x and multiplying by $u_{ttx}(t,x)$. We thus obtain

$$(3.10) \quad \frac{1}{2} \int_{-\infty}^{\infty} u_{ttx}^2(s,x)dx + \frac{1}{2} \int_{-\infty}^{\infty} \sigma'(v_x(s,x))u_{txx}^2(s,x)dx - \frac{1}{2} \int_{-\infty}^{\infty} u_{ttx}^2(0,x)dx$$

$$- \frac{1}{2} \int_{-\infty}^{\infty} \sigma'(v_x(0,x))u_{txx}^2(0,x)dx = - \int_0^s \int_{-\infty}^{\infty} k(0)u_{ttx}^2 dxdt$$

$$+ \int_0^s \int_{-\infty}^{\infty} \frac{1}{2}\sigma''(v_x)v_{tx}u_{txx}^2 dxdt + \int_0^s \int_{-\infty}^{\infty} \sigma''(v_x)v_{txx}u_{xx}u_{ttx}dxdt$$

$$+ \int_0^s \int_{-\infty}^{\infty} \sigma''(v_x)v_{tx}u_{ttx}u_{xxx}dxdt + \int_0^s \int_{-\infty}^{\infty} \sigma'''(v_x)v_{tx}v_{xx}u_{xx}u_{ttx}dxdt$$

$$+ \int_0^s \int_{-\infty}^{\infty} \phi_{tx}u_{ttx}dxdt - \int_0^s \int_{-\infty}^{\infty} k'(0)v_{tx}u_{ttx}dxdt - \int_0^s \int_{-\infty}^{\infty} (k''*v_{tx})u_{ttx}dxdt .$$

We now observe that $u_{tt}(0,x)$, $u_{ttx}(0,x)$ and $u_{ttt}(0,x)$ can be expressed through (3.5) in terms of $u_{0x}(x)$, $u_{0xx}(x)$, $u_{0xxx}(x)$, $u_1(x)$, $u_{1x}(x)$, $u_{1xx}(x)$, $\phi(0,x)$, $\phi_t(0,x)$ and $\phi_x(0,x)$. It follows that, since $u(t,x)$ is smooth, $u(t,x) \epsilon X(M,T)$ provided that M is sufficiently large and T is sufficiently small. We shall use (3.6)-(3.10) to estimate M and T independently of our provisional smoothness assumptions. To this end, using (3.3), (3.4) and the Cauchy-Schwarz inequality, we majorize each integral on the right-hand side of (3.6)-(3.10) by a fixed constant (i.e. independent of M) plus M^2T times a constant depending on M. For example, in (3.9),

$$\int_0^s \int_{-\infty}^{\infty} 2\sigma''(v_x)v_{tx}u_{ttt}u_{txx}dxdt \leq \max|\sigma''(v_x)v_{tx}| \int_0^s \int_{-\infty}^{\infty} (u_{ttt}^2 + u_{txx}^2)dxdt$$

$$\leq (M \max_{[-M,M]} |\sigma''(\cdot)|)M^2T ,$$

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$$\int_0^S \int_{-\infty}^{\infty} \phi_{tt} u_{ttt} dxdt \le \int_0^S \int_{-\infty}^{\infty} \phi_{tt}^2 dxdt + \int_0^S \int_{-\infty}^{\infty} u_{ttt}^2 dxdt \le \int_0^{\infty} \int_{-\infty}^{\infty} \phi_{tt}^2 dxdt + M^2 T \ ,$$

$$- \int_0^S \int_{-\infty}^{\infty} (k'' * v_{tt}) u_{ttt} dxdt \le \int_0^S \int_{-\infty}^{\infty} (k'' * v_{tt})^2 dxdt + \int_0^S \int_{-\infty}^{\infty} u_{ttt}^2 dxdt$$

$$\le [(\int_0^{\infty} |k''(t)| dt)^2 + 1] M^2 T \ .$$

Since $u_{xxx}(t,x)$ does not appear on the left-hand side of any of the estimates (3.6)-(3.10), we have to express it in terms of other derivatives via

(3.11) $$\sigma'(v_x) u_{xxx} = u_{ttx} + k(0) u_{tx} - \sigma''(v_x) v_{xx} u_{xx} - \phi_x + (k' * v_{tx})$$

which is obtained by differentiating (3.5) with respect to x. On account of $(\sigma)^*$, (3.3), (3.4) and in conjunction with (3.8), (3.10), equation (3.11) yields an upper estimate for $\int_{-\infty}^{\infty} u_{xxx}^2(t,x) dx$ by a fixed constant plus $M^2 T$ times a constant depending on M.

Combining all the above estimates we obtain

(3.12) $$\int_{-\infty}^{\infty} [u_t^2(t,x) + u_x^2(t,x) + u_{tt}^2(t,x) + u_{tx}^2(t,x) + u_{xx}^2(t,x) + u_{ttt}^2(t,x)$$

$$+ u_{ttx}^2(t,x) + u_{txx}^2(t,x) + u_{xxx}^2(t,x)] dx \le A(u_0, u_1, \phi) + B(M) M^2 T, \quad 0 \le t \le T \ .$$

The constant A can be estimated solely in terms of the $L^2(R)$-norms of u_{0x}, u_{0xx}, u_{0xxx}, u_1, u_{1x}, u_{1xx}; the $L^2([0,\infty); L^2(R))$ norms of ϕ, ϕ_t, ϕ_x, ϕ_{tt}, ϕ_{tx}; p_0 of $(\sigma)^*$; bounds on $\sigma'(\cdot)$, $\sigma''(\cdot)$ and $k'(\cdot)$. The constant B can be estimated solely in terms of M and bounds on $\sigma'(\cdot)$, $\sigma''(\cdot)$, $\sigma'''(\cdot)$, $k'(\cdot)$, $k''(\cdot)$. It follows that, even though (3.12) was established under supplementary smoothness conditions, its validity can be extended by a simple density argument to the standing assumptions of this section.

We now select $M \ge [2A(u_0, u_1, \phi)]^{1/2}$ and then $T \le [2B(M)]^{-1}$ in which case the right-hand side of (3.12) is dominated by M^2. It follows that with this selection of M and T, S maps $X(M,T)$ into itself. The proof of Lemma 3.1 is complete.

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We now equip $X(M,T)$ with the metric

$$(3.13) \qquad \rho(u,\bar{u}) = \max_{[0,T]} \{\int_{-\infty}^{\infty} [(u_t(t,x) - \bar{u}_t(t,x))^2 + (u_x(t,x) - \bar{u}_x(t,x))^2]dx\}^{1/2} .$$

Using the lower semicontinuity property of norms under weak convergence in Banach space, it is easily verified that $X(M,T)$ becomes a complete bounded metric space. We now have

Lemma 3.2. For M sufficiently large and T sufficiently small the map $S : X(M,T) \rightarrow X(M,T)$ is a contraction.

Proof. Let $v(t,x), \bar{v}(t,x) \in X(M,T)$. We set $u = Sv$, $\bar{u} = S\bar{v}$, $V = v - \bar{v}$, $U = u - \bar{u}$. Then $U(t,x)$ is a solution of the initial value problem

$$(3.14) \quad \begin{cases} U_{tt}(t,x) + k(0)U_t(t,x) = \sigma'(v_x(t,x))U_{xx}(t,x) + \chi(t,x)\bar{u}_{xx}(t,x)V_x(t,x) - (k'*V_t)(t,x), \\ \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad 0 \le t \le T, \quad x \in R \\ U(0,x) = 0, \ U_t(0,x) = 0, \quad x \in R \end{cases}$$

where $\chi(t,x)$ is the bounded continuous function

$$(3.15) \qquad \chi(t,x) = \begin{cases} \dfrac{\sigma'(v_x(t,x)) - \sigma'(\bar{v}_x(t,x))}{v_x(t,x) - \bar{v}_x(t,x)} & \text{if} \quad v_x(t,x) \ne \bar{v}_x(t,x) \\ \sigma''(v_x(t,x)) & \text{if} \quad v_x(t,x) = \bar{v}_x(t,x) . \end{cases}$$

Multiplying (3.14) by U_t, integrating over $[0,s] \times R$, $0 < s \le T$, and after an integration by parts we obtain

$$(3.16) \qquad \frac{1}{2}\int_{-\infty}^{\infty} U_t^2(s,x)dx + \frac{1}{2}\int_{-\infty}^{\infty} \sigma'(v_x(s,x))U_x^2(s,x)dx$$

$$= -\int_0^s \int_{-\infty}^{\infty} k(0)U_t^2 dxdt + \int_0^s \int_{-\infty}^{\infty} \frac{1}{2}\sigma''(v_x)v_{tx}U_x^2 dxdt$$

$$- \int_0^s \int_{-\infty}^{\infty} \sigma''(v_x)v_{xx}U_t U_x dxdt + \int_0^s \int_{-\infty}^{\infty} \chi\bar{u}_{xx}V_x U_t dxdt$$

$$- \int_0^s \int_{-\infty}^{\infty} (k'*V_t)U_t dxdt .$$

We minorize the left-hand side of (3.16), using $(\sigma)*$, and we majorize each integral on the right-hand side using (3.4) and the Cauchy-Schwarz inequality thus obtaining

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$$(3.17) \quad \int_{-\infty}^{\infty} [U_t^2(s,x) + U_x^2(s,x)]dx$$

$$\leq \; \max_{[0,T]} \int_{-\infty}^{\infty} [V_t^2(t,x) + V_x^2(t,x)]dxdt + \mu \int_0^s \int_{-\infty}^{\infty} [U_t^2(t,x) + U_x^2(t,x)]dxdt, \quad (0 \leq s \leq T)$$

where μ depends solely on M, $\max_{[-M,M]} |\sigma'(\cdot)|$, $\max_{[-M,M]} |\sigma''(\cdot)|$, p_0, $k(0)$, and $\int_0^{\infty} |k'(t)|dt$.

Applying Gronwall's lemma to (3.17) yields

$$(3.18) \quad \max_{[0,T]} \int_{-\infty}^{\infty} [U_t^2(t,x) + U_x^2(t,x)]dx \leq Te^{\mu T} \max_{[0,T]} \int_{-\infty}^{\infty} [V_t^2(t,x) + V_x^2(t,x)]dx .$$

We now fix M sufficiently large and, subsequently, we pick T so small that on the one hand S maps $X(M,T)$ into itself (Lemma 3.1) and on the other $Te^{\mu T} \leq \frac{1}{4}$. This implies that

$$(3.19) \qquad \rho(Sv, S\bar{v}) \leq \frac{1}{2} \rho(v, \bar{v}), \quad \text{for} \quad v, \bar{v} \in X(M,T) .$$

The proof of Lemma 3.2 is complete.

Proof of Theorem 3.1. From Lemma 3.2 and the Banach fixed point theorem we deduce the existence of a unique fixed point of S in $X(M,T)$, for some $M > 0$, $T > 0$, which will be a solution of (1.1) on $[0,T] \times R$. Let $T_0 \leq \infty$ be the maximal interval of existence of a solution $u(t,x)$ of (1.1) which satisfies (3.1) for all $T < T_0$. Then $u(t,x)$ is locally and hence also globally unique as the fixed point of a contraction.

If $T_0 < \infty$ and (3.2) is not satisfied, we can extend $u(t,x)$ up to $t = T_0$ so that $u(t,x) \in C^2([0,T_0] \times R)$ and, by weak convergence in $L^2(R)$, $u_x(T_0,x)$, $u_{xx}(T_0,x)$, $u_{xxx}(T_0,x)$, $u_t(T_0,x)$, $u_{tx}(T_0,x)$, $u_{txx}(T_0,x) \in L^2(R)$. But then $u(t,x)$ can be extended as a solution on a small interval $[T_0, T_0 + \epsilon]$ beyond T_0 which is a contradiction. This completes the proof of Theorem 3.1.

As $\sigma(\cdot)$, $k(t)$, and the data $u_0(x)$, $u_1(x)$ and $\phi(t,x)$ get smoother, the solution becomes smoother. We record below a regularity result that will be used in Sections 4 and 5.

Theorem 3.2. <u>Let the assumptions of Theorem 3.1 be satisfied. Under the additional</u>
<u>assumption</u>

$$(3.20) \quad \begin{cases} \sigma(\cdot) \in C^4(R) \\ u_{0xxxx}(x), u_{1xxx}(x) \in L^2(R) \\ \phi_{ttt}, \phi_{ttx}, \phi_{txx} \in L^2([0,\infty); L^2(R)) , \end{cases}$$

<u>the solution</u> $u(t,x)$ <u>of</u> (1.1), <u>established by Theorem 3.1, also has the property</u>

$$(3.21) \qquad u_{tttt}, u_{tttx}, u_{ttxx}, u_{txxx}, u_{xxxx} \in L^\infty([0,T]; L^2(R)) ,$$

<u>for every</u> $T < T_0$, <u>where</u> $[0,T_0)$ <u>is the maximal interval of existence.</u>

<u>Sketch of proof.</u> For positive $M, N, T,$ we denote by $Y(M,N,T)$ the set of functions
$u(t,x) \in C^3([0,T] \times R)$, with initial conditions $u(0,x) = u_0(x)$, $u_t(0,x) = u_1(x)$,
which satisfy (3.1), (3.3), (3.21) and

$$(3.22) \quad \sup_{[0,T]} \int_{-\infty}^{\infty} [u_t^2 + u_x^2 + u_{tt}^2 + u_{tx}^2 + u_{xx}^2 + u_{ttt}^2 + u_{ttx}^2 + u_{txx}^2 + u_{xxx}^2 + u_{tttt}^2$$
$$+ u_{tttx}^2 + u_{ttxx}^2 + u_{txxx}^2 + u_{xxxx}^2]dx \leq N^2 .$$

We consider again the map S that carries $v(t,x)$ to the solution $u(t,x)$ of (3.5)
and we try to show that it has a fixed point in $Y(M,N,T)$, for appropriate values of
M, N, T. We proceed as in the proof of Lemma 3.1. To the set of "energy" integrals
(3.6)-(3.10) we append another integral obtained by forming the third derivative of
(3.5), with respect to t, t and x, and then multiplying by u_{tttx}. This allows
the estimation of $\int_{-\infty}^{\infty} u_{tttx}^2(t,x)dx$ and $\int_{-\infty}^{\infty} u_{ttxx}^2(t,x)dx$. These estimates together with

(3.5) yield estimates for $\int_{-\infty}^{\infty} u_{tttt}^2(t,x)dx$, $\int_{-\infty}^{\infty} u_{txxx}^2(t,x)dx$ and $\int_{-\infty}^{\infty} u_{xxxx}^2(t,x)dx$.

Combining all above estimates we obtain, as in the proof of Lemma 3.1, the following
analog of (3.12):

$$(3.23) \quad \int_{-\infty}^{\infty} [u_t^2(t,x) + u_x^2(t,x) + u_{tt}^2(t,x) + u_{tx}^2(t,x) + u_{xx}^2(t,x) + u_{ttt}^2(t,x) + u_{ttx}^2(t,x)$$
$$+ u_{txx}^2(t,x) + u_{xxx}^2(t,x) + u_{tttt}^2(t,x) + u_{tttx}^2(t,x) + u_{ttxx}^2(t,x)$$
$$+ u_{txxx}^2(t,x) + u_{xxxx}^2(t,x)]dx \leq A^*(u_0,u_1,\phi) + B^*(M)N^2T .$$

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The crucial observation is that B^* depends on M but not on N! The reason is that in deriving (3.23) the integrals to be majorized are at most quadratic in the highest order derivatives (whence the term N^2) with coefficients that depend on derivatives of $v(t,x)$ of order at most two and are thus bounded, in view of (3.4), solely by functions of M. Similarly, we recall that μ, in (3.18), depends on M but not on N. It follows that the maximal interval of existence of the smoother solutions is controlled solely by M, i.e., (3.21) will be satisfied for any T for which (3.1) is satisfied. This concludes the sketch of the proof of Theorem 3.2.

4. Global Existence and Asymptotic Behavior of Solutions of the Heat Flow Equation. In this section we consider problem (HF) and we prove that if the initial data and the forcing term are "small" then the maximal interval of existence of the solution constructed in Section 3 is $[0,\infty)$ and that the solution decays to zero, as $t \to \infty$. Our strategy is to show that the dissipative mechanism induced by assumptions (a_H) overrides the growth tendencies of the solution caused by the nonlinearity of $\sigma(\cdot)$. Conclusion $(2.6)_H$ of Lemma 2.1 (iii), which was not used in Section 3, plays a crucial role in this argument. The precise form of the result is

Theorem 4.1. Let (σ), (a_H), (f) and (u_0) be satisfied. If the $L^2([0,\infty); L^2(R))$ norms of f, f_t, f_x, f_{tt}, f_{tx}, f_{xx}, f_{ttt}, f_{ttx}, f_{txx} and the $L^2(R)$ norms of u_{0x}, u_{0xx}, u_{0xxx} are sufficiently small, then there is a unique global solution $u(t,x) \in C^2([0,\infty) \times R)$ of (HF) and

(4.1) $\quad u_t, u_x, u_{tt}, u_{tx}, u_{xx}, u_{ttt}, u_{ttx}, u_{txx}, u_{xxx} \in L^\infty([0,\infty); L^2(R))$,

(4.2) $\quad u_t, u_{tt}, u_{tx}, u_{xx}, u_{ttt}, u_{ttx}, u_{txx}, u_{xxx} \in L^2([0,\infty); L^2(R))$,

(4.3) $\quad u_t(t,\cdot), u_{tt}(t,\cdot), u_{tx}(t,\cdot), u_{xx}(t,\cdot) \to 0, \ t \to \infty, \ \text{in } L^2(R)$,

(4.4) $\quad u_t(t,x), u_x(t,x), u_{tt}(t,x), u_{tx}(t,x), u_{xx}(t,x) \to 0, \ t \to \infty, \ \underline{\text{uniformly on}} \ R$.

Remark 4.1. It follows from the proof of Theorems 3.1 and 4.1 that the solution u of (HF) has a finite speed of propagation.

Proof. We will work with form (1.1) of problem (HF) with $\Phi(t,x)$ and $u_1(x)$ given by $(2.4)_H$ and $(2.5)_H$, respectively. In order to make sure that the problem is well-posed, we have to restrict the range of $u_x(t,x)$ to the set on which $\sigma' > 0$. To this end we introduce a constant $c_0 > 0$ such that

(4.5) $$\sigma'(w) \geq p_0 > 0, \quad w \in [-c_0, c_0] .$$

At this point it is convenient to define

(4.6) $$W(w) = \int_0^w \sigma(\xi) d\xi$$

and to note that, on account of (4.5),

(4.7) $$W(w) \geq \frac{1}{2} p_0 w^2, \quad w \in [-c_0, c_0] .$$

We will say that a quantity is "controllably small", if it can be made arbitrarily small by selecting the initial data and the forcing term appropriately small. For example, on account of (f), Lemma 2.1 (ii), $(2.4)_H$ and $(2.5)_H$, the $L^2([0,\infty); L^2(R))$ norms of ϕ, ϕ_t, ϕ_x, ϕ_{tt}, ϕ_{tx} and the $L^2(R)$ norms of u_1, u_{1x}, u_{1xx} are controllably small.

Our strategy is to show that there is a small positive number μ, $\mu < c_0$, depending solely upon α (the constant appearing in $(2.6)_H$), p_0, bounds of $|\sigma'(\cdot)|$, $|\sigma''(\cdot)|$ and $|\sigma'''(\cdot)|$ on $[-c_0, c_0]$, and the $L^1(0,\infty)$ norm of $k'(t)$, such that if the local solution $u(t,x)$ of (1.1), in the sense of Theorem 3.1, satisfies

(4.8) $\qquad |u_x(t,x)|, \ |u_{tx}(t,x)|, \ |u_{xx}(t,x)| \leq \mu, \ 0 \leq t < T, \ x \in R$,

then certain functionals of the solution are controllably small.

We begin by multiplying the equation in (1.1) by $u_t(t,x)$ and integrating each term over $[0,s] \times R$, $0 < s < T$. We integrate the term

$$\int_0^s \int_{-\infty}^\infty \sigma(u_x(t,x))_x u_t(t,x) dx dt$$

by parts with respect to x and use (4.6); we then use $(2.6)_H$ to estimate the term

$$\int_0^s \int_{-\infty}^\infty u_t(t,x) \frac{\partial}{\partial t} \int_0^t k(t-\tau) u_t(\tau,x) d\tau dx dt \ ,$$

and we thus obtain the estimate

(4.9) $\qquad \dfrac{1}{2} \displaystyle\int_{-\infty}^\infty u_t^2(s,x) dx + \int_{-\infty}^\infty W(u_x(s,x)) dx + \alpha \int_0^s \int_{-\infty}^\infty u_t^2 dx dt$

$$\leq \frac{1}{2} \int_{-\infty}^\infty u_t^2(0,x) dx + \int_{-\infty}^\infty W(u_x(0,x)) dx + \int_0^s \int_{-\infty}^\infty \phi u_t dx dt \ .$$

We note that each term on the right-hand side of (4.9) either is controllably small or can be majorized by the sum of a quantity that is controllably small and a quantity that is dominated by the dissipation term, such as

(4.10) $\qquad \displaystyle\int_0^s \int_{-\infty}^\infty \phi u_t dx dt \leq \frac{1}{\alpha} \int_0^s \int_{-\infty}^\infty \phi^2 dx dt + \frac{\alpha}{4} \int_0^s \int_{-\infty}^\infty u_t^2 dx dt \ .$

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Thus, for as long as (4.8) is satisfied with $\mu < c_0$, (4.9), (4.10), together with (4.7), yield that $\int_{-\infty}^{\infty} u_t^2(s,x)dx$, $\int_{-\infty}^{\infty} u_x^2(s,x)dx$ and $\int_0^s \int_{-\infty}^{\infty} u_t^2 dxdt$ are controllably small, uniformly on $[0,T]$.

We now derive two additional estimates, the first by differentiating the equation in (1.1) with respect to t and then multiplying by $u_{tt}(t,x)$, the second by differentiating (1.1) with respect to x and then multiplying by $u_{tx}(t,x)$. Following the procedure in the derivation of (4.9), and noting that

$$(4.11) \qquad \frac{\partial^2}{\partial t^2} (k*u_t)(t,x) = \frac{\partial}{\partial t} (k*u_{tt})(t,x) + k'(t)u_1(x) ,$$

we obtain

$$(4.12) \quad \frac{1}{2} \int_{-\infty}^{\infty} u_{tt}^2(s,x)dx + \frac{1}{2} \int_{-\infty}^{\infty} \sigma'(u_x(s,x))u_{tx}^2(s,x)dx + \alpha \int_0^s \int_{-\infty}^{\infty} u_{tt}^2 dxdt$$

$$\leq \frac{1}{2} \int_{-\infty}^{\infty} u_{tt}^2(0,x)dx + \frac{1}{2} \int_{-\infty}^{\infty} \sigma'(u_x(0,x))u_{tx}^2(0,x)dx + \int_0^s \int_{-\infty}^{\infty} \frac{1}{2} \sigma''(u_x)u_{tx}^3 dxdt$$

$$+ \int_0^s \int_{-\infty}^{\infty} \phi_t u_{tt} dxdt - \int_0^s \int_{-\infty}^{\infty} k'(t)u_1 u_{tt} dxdt ,$$

and

$$(4.13) \quad \frac{1}{2} \int_{-\infty}^{\infty} u_{tx}^2(s,x)dx + \frac{1}{2} \int_{-\infty}^{\infty} \sigma'(u_x(s,x))u_{xx}^2(s,x)dx + \alpha \int_0^s \int_{-\infty}^{\infty} u_{tx}^2 dxdt$$

$$\leq \frac{1}{2} \int_{-\infty}^{\infty} u_{tx}^2(0,x)dx + \frac{1}{2} \int_{-\infty}^{\infty} \sigma'(u_x(0,x))u_{xx}^2(0,x)dx + \int_0^s \int_{-\infty}^{\infty} \frac{1}{2} \sigma''(u_x)u_{tx}u_{xx}^2 dxdt$$

$$+ \int_0^s \int_{-\infty}^{\infty} \phi_x u_{tx} dxdt .$$

We add up (4.12), (4.13) and we claim that in the resulting inequality, and as long as (4.8) is satisfied with μ sufficiently small, each term on the right-hand side is either controllably small or can be majorized by the sum of a quantity that is controllably small and a quantity that is dominated by the dissipation term. Indeed, the $L^2(R)$ norm of $u_{tt}(0,x)$ is controllably small since

$$(4.14) \qquad u_{tt}(0,x) = \sigma(u_{0x}(x))_x + \phi(0,x) - k(0)u_1(x) ;$$

space-time integrals are majorized in the pattern of the following representative samples:

$$(4.15) \qquad \int_0^S \int_{-\infty}^{\infty} \phi_t u_{tt} dxdt \leq \frac{1}{\alpha} \int_0^{\infty} \int_{-\infty}^{\infty} \phi_t^2 dxdt + \frac{\alpha}{4} \int_0^{\infty} \int_{-\infty}^{\infty} u_{tt}^2 dxdt \ ,$$

$$(4.16) \quad -\int_0^S \int_{-\infty}^{\infty} k'(t) u_1 u_{tt} dxdt \leq \frac{1}{\alpha} \sup|k'(t)| \int_0^{\infty} |k'(t)| dt \int_{-\infty}^{\infty} u_1^2(x) dx + \frac{\alpha}{4} \int_0^S \int_{-\infty}^{\infty} u_{tt}^2 dxdt \ ,$$

$$(4.17) \qquad \int_0^S \int_{-\infty}^{\infty} \frac{1}{2} \sigma''(u_x) u_{tx}^3 dxdt \leq \frac{\mu}{2} \max_{[-c_0,c_0]} |\sigma''(\cdot)| \int_0^S \int_{-\infty}^{\infty} u_{tx}^2 dxdt \ ,$$

$$(4.18) \qquad \int_0^S \int_{-\infty}^{\infty} \frac{1}{2} \sigma''(u_x) u_{tx} u_{xx}^2 dxdt \leq \frac{\mu}{2} \max_{[-c_0,c_0]} |\sigma''(\cdot)| \int_0^S \int_{-\infty}^{\infty} u_{xx}^2 dxdt \ .$$

To estimate the integral on the right-hand side of (4.18), we express $u_{xx}(t,x)$ in terms of other derivatives by

$$(4.19) \qquad \sigma'(u_x(t,x)) u_{xx}(t,x) = u_{tt}(t,x) + k(0) u_t(t,x) + (k'*u_t)(t,x) - \Phi(t,x) \ ,$$

which yields

$$(4.20) \qquad p_0^2 \int_0^S \int_{-\infty}^{\infty} u_{xx}^2 dxdt \leq 4 \int_0^S \int_{-\infty}^{\infty} u_{tt}^2 dxdt + 4k^2(0) \int_0^S \int_{-\infty}^{\infty} u_t^2 dxdt$$

$$+ 4 \left(\int_0^{\infty} |k'(t)| dt\right)^2 \int_0^S \int_{-\infty}^{\infty} u_t^2 dxdt + 4 \int_0^S \int_{-\infty}^{\infty} \Phi^2 dxdt \ .$$

The restrictions imposed on μ are expressed in terms of parameters fixed a priori. For example, (4.17) imposes the restriction $\mu \max_{[-c_0,c_0]} |\sigma''(\cdot)| \leq \frac{\alpha}{4}$. The combination of (4.12), (4.13) and (4.20) yields that, as long as (4.8) is satisfied, $\int_{-\infty}^{\infty} u_{tt}^2(s,x) dx$,

$\int_{-\infty}^{\infty} u_{tx}^2(s,x) dx$, $\int_{-\infty}^{\infty} u_{xx}^2(s,x) dx$, $\int_0^S \int_{-\infty}^{\infty} u_{tt}^2 dxdt$, $\int_0^S \int_{-\infty}^{\infty} u_{tx}^2 dxdt$ and $\int_0^S \int_{-\infty}^{\infty} u_{xx}^2 dxdt$ are

controllably small, uniformly on $[0,T]$.

To get the final set of estimates we assume temporarily that the additional hypothesis (3.20) is satisfied. Thus, by Theorem 3.2, $u(t,x)$ enjoys the additional smoothness property (3.21). We form the second derivative of the equation in (1.1) with respect to t and we multiply by $u_{ttt}(t,x)$; also the second derivative of the equation in (1.1) with respect to t and x and we multiply by $u_{ttx}(t,x)$. Following

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the procedure used in the derivation of the previous estimates we obtain

$$(4.21) \quad \frac{1}{2} \int_{-\infty}^{\infty} u_{ttt}^2(s,x)dx + \frac{1}{2} \int_{-\infty}^{\infty} \sigma'(u_x(s,x))u_{ttx}^2(s,x)dx + \alpha \int_0^s \int_{-\infty}^{\infty} v_{ttt}^2 dxdt$$

$$\leq \frac{1}{2} \int_{-\infty}^{\infty} u_{ttt}^2(0,x)dx + \frac{1}{2} \int_{-\infty}^{\infty} \sigma'(u_x(0,x))u_{ttx}^2(0,x)dx + \int_0^s \int_{-\infty}^{\infty} \frac{1}{2} \sigma''(u_x)u_{tx}u_{ttx}^2 dxdt$$

$$+ \int_0^s \int_{-\infty}^{\infty} 2\sigma''(u_x)u_{tx}u_{tt}u_{txx} dxdt + \int_0^s \int_{-\infty}^{\infty} \sigma'''(u_x)u_{tx}^2 u_{xx}u_{ttt} dxdt$$

$$+ \int_0^s \int_{-\infty}^{\infty} \phi_{tt}u_{ttt} dxdt - \int_0^s \int_{-\infty}^{\infty} k'(t)u_{tt}(0,x)u_{ttt} dxdt - \int_0^s \int_{-\infty}^{\infty} k''(t)u_1 u_{ttt} dxdt \ ,$$

$$(4.22) \quad \frac{1}{2} \int_{-\infty}^{\infty} u_{ttx}^2(s,x)dx + \frac{1}{2} \int_{-\infty}^{\infty} \sigma'(u_x(s,x))u_{txx}^2(s,x)dx + \alpha \int_0^s \int_{-\infty}^{\infty} v_{ttx}^2 dxdt$$

$$\leq \frac{1}{2} \int_{-\infty}^{\infty} u_{ttx}^2(0,x)dx + \frac{1}{2} \int_{-\infty}^{\infty} \sigma'(u_x(0,x))u_{txx}^2(0,x)dx$$

$$+ \int_0^s \int_{-\infty}^{\infty} \frac{1}{2} \sigma''(u_x)u_{tx}u_{txx}^2 dxdt + \int_0^s \int_{-\infty}^{\infty} \sigma''(u_x)u_{xx}u_{ttx}u_{txx} dxdt$$

$$+ \int_0^s \int_{-\infty}^{\infty} \sigma''(u_x)u_{tx}u_{ttx}u_{xxx} dxdt + \int_0^s \int_{-\infty}^{\infty} \sigma'''(u_x)u_{tx}u_{xx}^2 u_{ttx} dxdt$$

$$+ \int_0^s \int_{-\infty}^{\infty} \phi_{tx}u_{ttx} dxdt - \int_0^s \int_{-\infty}^{\infty} k'(t)u_{1x}u_{ttx} dxdt \ .$$

We add (4.21), (4.22) and we claim that, as long as (4.8) is satisfied for a sufficiently small μ, then each term in the resulting inequality is either controllably small or is majorized by the sum of a quantity that is controllably small and a quantity that is dominated by the dissipation term. The $L^2(R)$ norms of $u_{ttt}(0,x)$ and $u_{ttx}(0,x)$ are controllably small since

$$(4.23) \quad u_{ttt}(0,x) = \sigma'(u_{0x}(x))u_{1xx}(x) + \sigma''(u_{0x}(x))u_{1x}(x)u_{0xx}(x) + \phi_t(0,x)$$
$$- k(0)u_{tt}(0,x) - k'(0)u_1(x) \ ,$$

$$(4.24) \quad u_{ttx}(0,x) = \sigma'(u_{0x}(x))u_{0xxx}(x) + \sigma''(u_{0x}(x))u_{0xx}^2(x) + \phi_x(0,x) - k(0)u_{1x}(x) \ .$$

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Whenever $u_{txx}(t,x)$ or $u_{xxx}(t,x)$ appear on the right-hand side, they should be estimated in terms of derivatives for which dissipation information is available with the help of

$$(4.25) \qquad \sigma'(u_x(t,x))u_{txx} = u_{ttt}(t,x) - \sigma''(u_x(t,x))u_{tx}(t,x)u_{xx}(t,x) - \phi_t(t,x)$$
$$+ k(0)u_{tt}(t,x) + k'(0)u_t(t,x) + (k''*u_t)(t,x) \ ,$$

$$(4.26) \qquad \sigma'(u_x(t,x))u_{xxx} = u_{ttx}(t,x) - \sigma''(u_x(t,x))u_{xx}^2(t,x) - \phi_x(t,x)$$
$$+ k(0)u_{tx}(t,x) + (k'*u_{tx})(t,x) \ .$$

Beyond that the estimations follow the usual pattern, e.g.,

$$(4.27) \qquad \int_0^s \int_{-\infty}^\infty \phi_{tt} u_{ttt} dx dt \le \frac{1}{\alpha} \int_0^s \int_{-\infty}^\infty \phi_{tt}^2 dx dt + \frac{\alpha}{4} \int_0^s \int_{-\infty}^\infty u_{ttt}^2 dx dt \ ,$$

$$(4.28) \quad \int_0^s \int_{-\infty}^\infty \sigma'''(u_x)u_{tx}^2 u_{xx} u_{ttt} dx dt \le \frac{\mu^2}{2} \max_{[-c_0, c_0]} |\sigma'''(\cdot)| \int_0^s \int_{-\infty}^\infty (u_{tx}^2 + u_{ttt}^2) dx dt \ ,$$

$$(4.29) \quad - \int_0^s \int_{-\infty}^\infty k''(t)u_1 u_{ttt} dx dt \le \frac{1}{\alpha} \sup |k''(t)| \int_0^\infty |k''(t)| dt \int_{-\infty}^\infty u_1^2(x) dx + \frac{\alpha}{4} \int_0^s \int_{-\infty}^\infty u_{ttt}^2 dx dt \ .$$

Combining (4.21), (4.22), (4.25) and (4.26) we deduce that $\int_{-\infty}^\infty u_{ttt}^2(s,x) dx$,

$\int_{-\infty}^\infty u_{ttx}^2(s,x) dx$, $\int_{-\infty}^\infty u_{txx}^2(s,x) dx$, $\int_{-\infty}^\infty u_{xxx}^2(s,x) dx$, $\int_0^s \int_{-\infty}^\infty u_{ttt}^2 dx dt$, $\int_0^s \int_{-\infty}^\infty u_{ttx}^2 dx dt$,

$\int_0^s \int_{-\infty}^\infty u_{txx}^2 dx dt$ and $\int_0^s \int_{-\infty}^\infty u_{xxx}^2 dx dt$ are controllably small, uniformly on $[0,T)$. Moreover, we observe that the estimates involved depend solely on parameters depending on the standing assumptions of the theorem and not on the additional assumptions (3.20). Therefore, by a straightforward density argument, we may remove these extraneous assumptions.

We now put together the information we have collected on controllably small quantities in all of our estimates, and we select initial data and forcing term so "small" that, as long as (4.8) is satisfied,

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(4.30) $\int_{-\infty}^{\infty} [u_t^2(s,x) + u_x^2(s,x) + u_{tt}^2(s,x) + u_{tx}^2(s,x) + u_{xx}^2(s,x) + u_{ttt}^2(s,x)$

$+ u_{ttx}^2(s,x) + u_{txx}^2(s,x) + u_{xxx}^2(s,x)]dx$

$+ \int_0^s \int_{-\infty}^{\infty} [u_t^2 + u_{tt}^2 + u_{tx}^2 + u_{xx}^2 + u_{ttt}^2 + u_{ttx}^2 + u_{txx}^2 + u_{xxx}^2]dxdt \leq \mu^2 ,$

$0 \leq s < T$. However, (4.30) implies, in return, (4.8) and the cycle closes. Once (4.30) has been established, Theorem 3.1 yields that the maximal interval of existence of $u(t,x)$ is $[0,\infty)$ and (4.30) is satisfied for $0 \leq s < \infty$. In particular, (4.1) and (4.2) are satisfied.

Statement (4.3) is an immediate corollary of (4.1), (4.2) while (4.4) is a corollary of (4.3) and (4.1). The proof is complete.

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5. Global Existence and Asymptotic Behavior of Solutions of the Viscoelasticity

Equation. We show here that when the initial data and the forcing term are "small" problem (VE) admits a unique, globally defined, solution which decays to zero as $t \to \infty$. We note that the solution u of (VE) has a finite speed of propagation (see Remark 4.1). The estimate (2.11) of Lemma 2.3 (iii) plays a crucial role in the analysis. The result is given by

Theorem 5.1. Let (σ), (a_V), (g), (u_0) and (u_1) be satisfied. If the $L^1([0,\infty); L^2(R))$ norms of g, g_t, the $L^2([0,\infty); L^2(R))$ norms of g_x, g_{tt}, g_{tx} and the $L^2(R)$ norms of u_{0x}, u_{0xx}, u_{0xxx}, u_1, u_{1x}, u_{1xx} are sufficiently small, then there is a unique global solution $u(t,x) \in C^2([0,\infty) \times R)$ of (VE) and

(5.1) $\qquad u_t, u_x, u_{tt}, u_{tx}, u_{xx}, u_{ttt}, u_{ttx}, u_{txx}, u_{xxx} \in L^\infty([0,\infty); L^2(R))$,

(5.2) $\qquad u_{tt}, u_{tx}, u_{xx}, u_{ttt}, u_{ttx}, u_{txx}, u_{xxx} \in L^2([0,\infty); L^2(R))$

(5.3) $\qquad u_{tt}(t,\cdot), u_{tx}(t,\cdot), u_{xx}(t,\cdot) \to 0, \ t \to \infty, \ \text{in} \ L^2(R)$

(5.4) $\quad u_t(t,x), u_x(t,x), u_{tt}(t,x), u_{tx}(t,x), u_{xx}(t,x) \to 0, \ t \to \infty, \ \text{unif. on} \ R$.

Proof. As in the proof of Theorem 4.1, we will work here with form (1.1) of our problem. We introduce again c_0 by (4.5) and $W(w)$ by (4.6) noting (4.7). We again consider a local solution $u(t,x)$ of (1.1), in the sense of Theorem 3.1, which satisfies (4.8) for some T, $0 < T \le \infty$, and a small positive μ, $\mu < c_0$, to be specified later.

To get the first estimate, we multiply the equation in (1.1) by $u_t(t,x)$, we integrate over $[0,s] \times R$, $0 < s < T$, integrating by parts with respect to x, and we use $(2.6)_V$ thus obtaining

(5.5) $\quad \dfrac{1}{2} \displaystyle\int_{-\infty}^{\infty} u_t^2(s,x)dx + \int_{-\infty}^{\infty} W(u_x(s,x))dx \le \dfrac{1}{2} \int_{-\infty}^{\infty} u_t^2(0,x)dx + \int_{-\infty}^{\infty} W(u_x(0,x))dx + \int_0^s \int_{-\infty}^{\infty} \Phi u_t \, dxdt$.

Noting that

(5.6) $\quad \displaystyle\int_0^s \int_{-\infty}^{\infty} \Phi u_t \, dxdt \le \dfrac{1}{4} \max_{[0,s]} \int_{-\infty}^{\infty} u_t^2(t,x)dx + \left(\int_0^\infty \left(\int_{-\infty}^{\infty} \Phi^2 dx \right)^{1/2} dt \right)^2$

and using (4.7), we deduce from (5.5)

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(5.7) $\frac{1}{2} \int_{-\infty}^{\infty} u_t^2(s,x)dx + p_0 \int_{-\infty}^{\infty} u_x^2(s,x)dx$

$$\leq \int_{-\infty}^{\infty} u_1^2(x)dx + 2 \int_{-\infty}^{\infty} W(u_{0x}(x))dx + 2(\int_0^{\infty} (\int_{-\infty}^{\infty} \phi^2 dx)^{1/2} dt)^2, \quad 0 \leq s < T .$$

It is easily seen that, on account of $(2.4)_V$, the $L^1([0,\infty); L^2(R))$ norm of ϕ is controllably small, so that it follows from (5.7) that $\int_{-\infty}^{\infty} u_t^2(s,x)dx$ and $\int_{-\infty}^{\infty} u_x^2(s,x)dx$ are controllably small, uniformly on $[0,T)$.

In the following estimates we seek to take advantage of the dissipative mechanism which manifests itself through Lemma 2.3 (iii). To this end, we shall use the equivalent form (2.9) of our problem (VE).

We differentiate (2.9) with respect to t and we multiply the resulting equation, first by $u_{tt}(t,x)$ and then by $u_t(t,x)$. Integrating over $[0,s] \times R, 0 < s < T$, integrating by parts with respect to x, etc., we end up with the following two equations:

(5.8) $\frac{1}{2} \int_{-\infty}^{\infty} u_{tt}^2(s,x)dx + \frac{1}{2} \int_{-\infty}^{\infty} \sigma'(u_x(s,x))u_{tx}^2(s,x)dx + \int_0^s \int_{-\infty}^{\infty} u_{tt}(r*u_{tt})_t dxdt$

$$- \beta \int_0^s \int_{-\infty}^{\infty} \sigma'(u_x)u_{tx}^2 dxdt = \frac{1}{2} \int_{-\infty}^{\infty} u_{tt}^2(0,x)dx + \frac{1}{2} \int_{-\infty}^{\infty} \sigma'(u_x(0,x))u_{tx}^2(0,x)dx$$

$$+ \int_0^s \int_{-\infty}^{\infty} \frac{1}{2} \sigma''(u_x)u_{tx}^3 dxdt - \beta \int_{-\infty}^{\infty} \sigma(u_x(s,x))u_{tx}(s,x)dx + \beta \int_{-\infty}^{\infty} \sigma(u_x(0,x))u_{tx}(0,x)dx$$

$$+ \int_0^s \int_{-\infty}^{\infty} \Psi_t u_{tt} dxdt ;$$

(5.9) $\frac{\beta}{2a_\infty} \int_{-\infty}^{\infty} u_t^2(s,x)dx + \beta \int_{-\infty}^{\infty} W(u_x(s,x))dx - \int_0^s \int_{-\infty}^{\infty} u_{tt}(R*u_{tt})dxdt$

$$- \int_0^s \int_{-\infty}^{\infty} u_{tt}^2 dxdt + \int_0^s \int_{-\infty}^{\infty} \sigma'(u_x)u_{tx}^2 dxdt = \frac{\beta}{2a_\infty} \int_{-\infty}^{\infty} u_t^2(0,x)dx + \beta \int_{-\infty}^{\infty} W(u_x(0,x))dx$$

$$- \int_{-\infty}^{\infty} u_t(s,x)u_{tt}(s,x)dx + \int_{-\infty}^{\infty} u_t(0,x)u_{tt}(0,x)dx$$

$$- \int_{-\infty}^{\infty} u_t(s,x)(R*u_{tt})(s,x)dx + \int_0^s \int_{-\infty}^{\infty} \Psi_t u_t dxdt .$$

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We now multiply (5.8) by q (see Lemma 2.3) and we add it to (5.9). Using (2.11) we obtain

(5.10) $\quad \dfrac{\beta}{2a_\infty} \displaystyle\int_{-\infty}^{\infty} u_t^2(s,x)dx + \beta \int_{-\infty}^{\infty} W(u_x(s,x))dx + \dfrac{q}{2}\int_{-\infty}^{\infty} u_{tt}^2(s,x)dx + \dfrac{q}{2}\int_{-\infty}^{\infty} \sigma'(u_x(s,x))u_{tx}^2(s,x)dx$

$\qquad + \gamma \displaystyle\int_0^s \int_{-\infty}^{\infty} u_{tt}^2 dxdt + (1-q\beta)\int_0^s \int_{-\infty}^{\infty} \sigma'(u_x)u_{tx}^2 dxdt$

$\qquad \leq \dfrac{\beta}{2a_\infty} \displaystyle\int_{-\infty}^{\infty} u_t^2(0,x)dx + \beta\int_{-\infty}^{\infty} W(u_x(0,x))dx + \dfrac{q}{2}\int_{-\infty}^{\infty} u_{tt}^2(0,x)dx$

$\qquad + \dfrac{q}{2} \displaystyle\int_{-\infty}^{\infty} \sigma'(u_x(0,x))u_{tx}^2(0,x)dx - \int_{-\infty}^{\infty} u_t(s,x)u_{tt}(s,x)dx + \int_{-\infty}^{\infty} u_t(0,x)u_{tt}(0,x)dx$

$\qquad - q\beta \displaystyle\int_{-\infty}^{\infty} \sigma(u_x(s,x))u_{tx}(s,x)dx + q\beta \int_{-\infty}^{\infty} \sigma(u_x(0,x))u_{tx}(0,x)dx$

$\qquad - \displaystyle\int_{-\infty}^{\infty} u_t(s,x)(R*u_{tt})(s,x)dx + q\int_0^s \int_{-\infty}^{\infty} \dfrac{1}{2}\sigma''(u_x)u_{tx}^3 dxdt$

$\qquad + \displaystyle\int_0^s \int_{-\infty}^{\infty} \Psi_t u_t dxdt + q\int_0^s \int_{-\infty}^{\infty} \Psi_t u_{tt} dxdt .$

We claim that each term on the right-hand side of (5.10) is either controllably small, as long as (4.8) is satisfied with sufficiently small μ, or it can be majorized by the sum of a controllably small quantity and a quantity dominated by the left-hand side of (5.10). Thus, for example, the $L^2(R)$ norm of $u_{tt}(0,x)$ is controllably small in view of (4.14). Also

(5.11) $\quad -\displaystyle\int_{-\infty}^{\infty} u_t(s,x)u_{tt}(s,x)dx \leq \dfrac{1}{q}\int_{-\infty}^{\infty} u_t^2(s,x)dx + \dfrac{q}{4}\int_{-\infty}^{\infty} u_{tt}^2(s,x)dx ,$

(5.12) $\quad -\displaystyle\int_{-\infty}^{\infty} u_t(s,x)(R*u_{tt})(s,x)dx \leq \int_{-\infty}^{\infty} |u_t(s,x)| \{\int_0^s R^2(\tau)d\tau\}^{1/2}\{\int_0^s u_{tt}^2(\tau,x)d\tau\}^{1/2}dx$

$\qquad\qquad \leq \dfrac{1}{\gamma} \sup|R(t)| \displaystyle\int_0^\infty |R(t)|dt \int_{-\infty}^{\infty} u_t^2(s,x)dx + \dfrac{\gamma}{4}\int_0^s \int_{-\infty}^{\infty} u_{tt}^2 dxdt ,$

(5.13) $\quad q\displaystyle\int_0^s \int_{-\infty}^{\infty} \dfrac{1}{2}\sigma''(u_x)u_{tx}^3 dxdt \leq \dfrac{\mu q}{2} \max_{[-c_0,c_0]} |\sigma''(\cdot)| \int_0^s \int_{-\infty}^{\infty} u_{tx}^2 dxdt ,$

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(5.14) $$\int_0^s \int_{-\infty}^{\infty} \Psi_t u_t \, dx dt \leq \frac{1}{4} \max_{[0,s]} \int_{-\infty}^{\infty} u_t^2(t,x) dx + \left(\int_0^s \left(\int_{-\infty}^{\infty} \Psi_t^2 dx \right)^{1/2} dt \right)^2 ,$$

(5.15) $$q \int_0^s \int_{-\infty}^{\infty} \Psi_t u_{tt} \, dx dt \leq \frac{q^2}{\gamma} \int_0^s \int_{-\infty}^{\infty} \Psi_t^2 dx dt + \frac{\gamma}{4} \int_0^s \int_{-\infty}^{\infty} u_{tt}^2 dx dt .$$

After the above estimations (5.10) yields that $\int_{-\infty}^{\infty} u_{tt}^2(s,x) dx$, $\int_{-\infty}^{\infty} u_{tx}^2(s,x) dx$, $\int_0^s \int_{-\infty}^{\infty} u_{tt}^2 dx dt$ and $\int_0^s \int_{-\infty}^{\infty} u_{tx}^2 dx dt$ are controllably small, uniformly on $[0,T]$, of course always as long as (4.8) is satisfied for a sufficiently small μ (from (5.13), $\mu q \max_{[-c_0, c_0]} |\sigma''(\cdot)| \leq (1 - q\beta)p_0$). Using (4.19) which we rewrite, after an integration by parts, in the form

(5.16) $$\sigma'(u_x(t,x))u_{xx}(t,x) = u_{tt}(t,x) + k(t)u_1(x) + (k*u_{tt})(t,x) - \Phi(t,x) ,$$

we deduce that also $\int_{-\infty}^{\infty} u_{xx}^2(s,x) dx$ and $\int_0^s \int_{-\infty}^{\infty} u_{xx}^2 dx dt$ are controllably small, uniformly on $[0,T]$.

To get the next estimate we assume temporarily that condition (3.20) holds, so that $u(t,x)$ is smoother, we take the second derivative of (2.9) with respect to t and x and we then multiply the resulting equation first by $u_{ttx}(t,x)$ and then by $u_{tx}(t,x)$. We integrate over $[0,s] \times R$, $0 < s < T$, and after a long computation we arrive at the following two equations:

(5.17) $$\frac{1}{2} \int_{-\infty}^{\infty} u_{ttx}^2(s,x) dx + \frac{1}{2} \int_{-\infty}^{\infty} \sigma'(u_x(s,x)) u_{txx}^2(s,x) dx + \int_0^s \int_{-\infty}^{\infty} u_{ttx}(r*u_{ttx})_t \, dx dt$$

$$- \beta \int_0^s \int_{-\infty}^{\infty} \sigma'(u_x) u_{txx}^2 dx dt = \frac{1}{2} \int_{-\infty}^{\infty} u_{ttx}^2(0,x) dx + \frac{1}{2} \int_{-\infty}^{\infty} \sigma'(u_x(0,x)) u_{txx}^2(0,x) dx$$

$$- \beta \int_{-\infty}^{\infty} \sigma'(u_x(s,x)) u_{xx}(s,x) u_{txx}(s,x) dx + \beta \int_{-\infty}^{\infty} \sigma'(u_x(0,x)) u_{xx}(0,x) u_{txx}(0,x) dx$$

$$+ \int_0^s \int_{-\infty}^{\infty} \frac{1}{2} \sigma''(u_x) u_{tx} u_{txx}^2 dx dt + \int_0^s \int_{-\infty}^{\infty} \sigma''(u_x) u_{xx} u_{ttx} u_{txx} dx dt$$

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$$- \int_0^s \int_{-\infty}^\infty \sigma''(u_x)u_{tx}u_{ttx}u_{xxx}dxdt - \int_0^s \int_{-\infty}^\infty \sigma'''(u_x)u_{tx}u_{xx}^2 u_{ttx}dxdt$$

$$+ \beta \int_0^s \int_{-\infty}^\infty \sigma''(u_x)u_{tx}u_{xx}u_{txx}dxdt + \int_0^s \int_{-\infty}^\infty \Psi_{tx}u_{ttx}dxdt \ ,$$

$$(5.18) \quad \frac{\beta}{2a_\infty}\int_{-\infty}^\infty u_{tx}^2(s,x)dx + \frac{\beta}{2}\int_{-\infty}^\infty \sigma'(u_x(s,x))u_{xx}^2(s,x)dx - \int_0^s \int_{-\infty}^\infty u_{ttx}(R*u_{ttx})dxdt$$

$$- \int_0^s \int_{-\infty}^\infty u_{ttx}^2 dxdt + \int_0^s \int_{-\infty}^\infty \sigma'(u_x)u_{txx}^2 dxdt = \frac{\beta}{2a_\infty}\int_{-\infty}^\infty u_{tx}^2(0,x)dx$$

$$+ \frac{\beta}{2}\int_{-\infty}^\infty \sigma'(u_x(0,x))u_{xx}^2(0,x)dx - \int_{-\infty}^\infty u_{tx}(s,x)u_{ttx}(s,x)dx + \int_{-\infty}^\infty u_{tx}(0,x)u_{ttx}(0,x)dx$$

$$- \int_{-\infty}^\infty u_{tx}(s,x)(R*u_{ttx})(s,x)dx - \int_0^s \int_{-\infty}^\infty \sigma''(u_x)u_{tx}u_{xx}u_{txx}dxdt$$

$$+ \frac{\beta}{2}\int_0^s \int_{-\infty}^\infty \sigma''(u_x)u_{tx}u_{xx}^2 dxdt + \int_0^s \int_{-\infty}^\infty \Psi_{tx}u_{tx}dxdt \ .$$

Multiplying (5.17) by q then adding it to (5.18) and using (2.11), we obtain

$$(5.19) \quad \frac{\beta}{2a_\infty}\int_{-\infty}^\infty u_{tx}^2(s,x)dx + \frac{\beta}{2}\int_{-\infty}^\infty \sigma'(u_x(s,x))u_{xx}^2(s,x)dx + \frac{q}{2}\int_{-\infty}^\infty u_{ttx}^2(s,x)dx$$

$$+ \frac{q}{2}\int_{-\infty}^\infty \sigma'(u_x(s,x))u_{txx}^2(s,x)dx + \gamma \int_0^s \int_{-\infty}^\infty u_{ttx}^2 dxdt$$

$$+ (1 - q\beta)\int_0^s \int_{-\infty}^\infty \sigma'(u_x)u_{txx}^2 dxdt \leq \frac{\beta}{2a_\infty}\int_{-\infty}^\infty u_{tx}^2(0,x)dx + \frac{\beta}{2}\int_{-\infty}^\infty \sigma'(u_x(0,x))u_{xx}^2(0,x)dx$$

$$+ \frac{q}{2}\int_{-\infty}^\infty u_{ttx}^2(0,x)dx + \frac{q}{2}\int_{-\infty}^\infty \sigma'(u_x(0,x))u_{txx}^2(0,x)dx - \int_{-\infty}^\infty u_{tx}(s,x)u_{ttx}(s,x)dx$$

$$+ \int_{-\infty}^\infty u_{tx}(0,x)u_{ttx}(0,x)dx - q\beta \int_{-\infty}^\infty \sigma'(u_x(s,x))u_{xx}(s,x)u_{txx}(s,x)dx$$

$$+ q\beta \int_{-\infty}^\infty \sigma'(u_x(0,x))u_{xx}(0,x)u_{txx}(0,x)dx - \int_{-\infty}^\infty u_{tx}(s,x)(R*u_{ttx})(s,x)dx$$

$$+ q \int_0^s \int_{-\infty}^{\infty} \frac{1}{2} \sigma''(u_x) u_{tx} u_{txx}^2 \, dxdt + q \int_0^s \int_{-\infty}^{\infty} \sigma''(u_x) u_{xx} u_{ttx} u_{txx} \, dxdt$$

$$- q \int_0^s \int_{-\infty}^{\infty} \sigma''(u_x) u_{tx} u_{ttx} u_{xxx} \, dxdt - q \int_0^s \int_{-\infty}^{\infty} \sigma'''(u_x) u_{tx} u_{xx}^2 u_{ttx} \, dxdt$$

$$+ \beta q \int_0^s \int_{-\infty}^{\infty} \sigma''(u_x) u_{tx} u_{xx} u_{txx} \, dxdt - \int_0^s \int_{-\infty}^{\infty} \sigma''(u_x) u_{tx} u_{xx} u_{txx} \, dxdt$$

$$+ \frac{\beta}{2} \int_0^s \int_{-\infty}^{\infty} \sigma''(u_x) u_{tx} u_{xx}^2 \, dxdt + \int_0^s \int_{-\infty}^{\infty} \Psi_{tx} u_{tx} \, dxdt + q \int_0^s \int_{-\infty}^{\infty} \Psi_{tx} u_{ttx} \, dxdt \ .$$

As long as (4.8) is satisfied with μ sufficiently small, each term on the right-hand side of (5.19) is either controllably small or it can be estimated by the sum of a controllably small quantity and a quantity dominated by the left-hand side of (5.19). To show that the $L^2(R)$ norm of $u_{ttx}(0,x)$ is controllably small, we use (4.24). To estimate u_{xxx} we express it, with the help of (4.26), in terms of derivatives on which we already have information. The remaining steps of the estimation follow the by now familiar pattern. For example,

$$(5.20) \qquad - \int_{-\infty}^{\infty} u_{tx}(s,x) u_{ttx}(s,x) \, dx \leq \frac{1}{q} \int_{-\infty}^{\infty} u_{tx}^2(s,x) \, dx + \frac{q}{4} \int_{-\infty}^{\infty} u_{ttx}^2(s,x) \, dx \ ,$$

$$(5.21) \qquad - \int_{-\infty}^{\infty} u_{tx}(s,x) (R * u_{ttx})(s,x) \, dx$$

$$\leq \frac{1}{\gamma} \sup |R(t)| \int_0^{\infty} |R(t)| \, dt \int_{-\infty}^{\infty} u_{tx}^2(s,x) \, dx + \frac{\gamma}{4} \int_0^s \int_{-\infty}^{\infty} u_{ttx}^2 \, dxdt \ ,$$

$$(5.22) \qquad q \int_0^s \int_{-\infty}^{\infty} \sigma''(u_x) u_{xx} u_{ttx} u_{txx} \, dxdt \leq \frac{\mu q}{2} \max_{[-c_0, c_0]} |\sigma''(\cdot)| \int_0^s \int_{-\infty}^{\infty} (u_{ttx}^2 + u_{txx}^2) \, dxdt \ ,$$

$$(5.23) \qquad q \int_0^s \int_{-\infty}^{\infty} \Psi_{tx} u_{ttx} \, dxdt \leq \frac{q^2}{\gamma} \int_0^s \int_{-\infty}^{\infty} \Psi_{tx}^2 \, dxdt + \frac{\gamma}{4} \int_0^s \int_{-\infty}^{\infty} u_{ttx}^2 \, dxdt \ .$$

Thus, (5.19) yields that, as long as (4.8) is satisfied with small μ, $\int_{-\infty}^{\infty} u_{ttx}^2(s,x) \, dx$, $\int_{-\infty}^{\infty} u_{txx}^2(s,x) \, dx$, $\int_0^s \int_{-\infty}^{\infty} u_{ttx}^2 \, dxdt$ and $\int_0^s \int_{-\infty}^{\infty} u_{txx}^2 \, dxdt$ are controllably small, uniformly

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on [0,T]. We now discard, with the help of a density argument, the extraneous

hypothesis (3.20). Using again (4.26) we conclude that $\int_{-\infty}^{\infty} u^2_{xxx}(s,x)dx$ and

$\int_0^s \int_{-\infty}^{\infty} u^2_{xxx}dxdt$ are controllably small, uniformly on [0,T]. Finally, using (4.25),

after rewriting it in the form

$$(5.24) \qquad u_{ttt}(t,x) = \sigma'(u_x(t,x))u_{txx}(t,x) + \sigma''(u_x(t,x))u_{tx}(t,x)u_{xx}(t,x)$$
$$+ \phi_t(t,x) - k(0)u_{tt}(t,x) - k'(t)u_1(x) - (k'*u_{tt})(t,x) \ ,$$

we deduce that $\int_{-\infty}^{\infty} u^2_{ttt}(s,x)$ and $\int_0^s \int_{-\infty}^{\infty} u^2_{ttt}dxdt$ are also controllably small, uniformly

on [0,T].

Combining the above information, we select initial data and forcing term so small

that, as long as (4.8) is satisfied,

$$(5.25) \qquad \int_{-\infty}^{\infty} [u^2_t(s,x) + u^2_x(s,x) + u^2_{tt}(s,x) + u^2_{tx}(s,x) + u^2_{xx}(s,x) + u^2_{ttt}(s,x)$$
$$+ u^2_{ttx}(s,x) + u^2_{txx}(s,x) + u^2_{xxx}(s,x)]dx$$
$$+ \int_0^s \int_{-\infty}^{\infty} [u^2_{tt} + u^2_{tx} + u^2_{xx} + u^2_{ttt} + u^2_{ttx} + u^2_{txx} + u^2_{xxx}]dxdt \le \mu^2 \ ,$$

$0 \le s < T$. Since, in return, (5.25) implies (4.8), we conclude with the help of

Theorem 3.1 that the maximal interval of existence of $u(t,x)$ is $[0,\infty)$ and that

(5.25) is satisfied for $s \epsilon [0,\infty)$. In particular, (5.1), (5.2) hold. Assertion (5.3)

follows from (5.1), (5.2) and (5.4) is a corollary of (5.1) and (5.3). The proof is

complete.

Remark 5.1. If $a(t) = \frac{1}{2} (1 + e^{-t})$, (VE) is easily shown to be equivalent to the

Cauchy problem

$$\begin{cases} u_{ttt} + u_{tt} = \sigma(u_x)_{tx} + \frac{1}{2} \sigma(u_x)_x + g + g_t \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x), \ u_{tt}(0,x) = \sigma(u_{0x}(x))_x \end{cases}$$

that was studied by J. Greenberg [3]. We note that our Theorem 5.1 applies to this

problem.

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6. **Boundary Value Problems.** In this section we make a few remarks on initial-boundary value problems for the heat flow and the viscoelasticity equations. We assume that the configuration of the body is the interval $[0,1]$ and we impose homogeneous boundary conditions of the Neumann type. For the heat flow equation these boundary conditions mean that the boundary is thermally insulated, while for the viscoelasticity equation they mean that the end points are free. In the place of (HF) and (VE) we now have

$$(HF)^* \quad \begin{cases} u_t(t,x) = \int_0^t a(t-\tau)\sigma(u_x(\tau,x))_x d\tau + f(t,x), & 0 < t < \infty, \ 0 < x < 1, \\ u(0,x) = u_0(x), & 0 \le x \le 1, \\ u_x(t,0) = u_x(t,1) = 0, & 0 < t < \infty, \end{cases}$$

$$(VE)^* \quad \begin{cases} u_{tt}(t,x) = \sigma(u_x(t,x))_x + \int_0^t a'(t-\tau)\sigma(u_x(\tau,x))_x d\tau + g(t,x), & 0 < t < \infty, \ 0 < x < 1, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & 0 \le x \le 1, \\ u_x(t,0) = u_x(t,1) = 0, & 0 < t < \infty. \end{cases}$$

The proper replacements of assumptions (f), (g), (u_0) and (u_1) are

$$(f)^* \qquad f, f_t, f_x, f_{tt}, f_{tx}, f_{xx}, f_{ttt}, f_{ttx}, f_{txx} \in L^2([0,\infty); L^2(0,1)),$$

$$(g)^* \qquad g, g_t \in L^1([0,\infty); L^2(0,1)), g_x, g_{tt}, g_{tx} \in L^2([0,\infty); L^2(0,1)),$$

$$(u_0)^* \qquad \begin{cases} u_{0x}, u_{0xx}, u_{0xxx} \in L^2(0,1) \\ u_{0x}(0) = u_{0x}(1) = 0 \end{cases}$$

$$(u_1)^* \qquad \begin{cases} u_1, u_{1x}, u_{1xx} \in L^2(0,1) \\ u_{1x}(0) = u_{1x}(1) = 0 \end{cases}$$

For problems $(HF)^*$ and $(VE)^*$ propositions analogous to Theorems 4.1 and 5.1 hold, namely,

Theorem 6.1. Let (σ), (a_H), $(f)^*$ and $(u_0)^*$ be satisfied. If the $L^2([0,\infty); L^2(0,1))$ norms of $f, f_t, f_x, f_{tt}, f_{tx}, f_{xx}, f_{ttt}, f_{ttx}, f_{txx}$ and the $L^2(R)$ norms of $u_{0x}, u_{0xx}, u_{0xxx}$ are sufficiently small, then there is a unique global solution

$u(t,x) \in C^2([0,\infty) \times [0,1])$ $\underline{of}$ (HF)* $\underline{and}$

(6.1) $\qquad\qquad u_t, u_x, u_{tt}, u_{tx}, u_{xx}, u_{ttt}, u_{ttx}, u_{txx}, u_{xxx} \in L^\infty([0,\infty); L^2(0,1))$,

(6.2) $\qquad\qquad u_t, u_{tt}, u_{tx}, u_{xx}, u_{ttt}, u_{ttx}, u_{txx}, u_{xxx} \in L^2([0,\infty); L^2(0,1))$.

(6.3) $\quad u_t(t,x), u_x(t,x), u_{tt}(t,x), u_{tx}(t,x), u_{xx}(t,x) \to 0, \; t \to \infty, \; \underline{unif.} \; \underline{on} \; (0,1)$.

$\underline{\text{Theorem 6.2.}}$ $\underline{Let}$ (σ), (a_v), $(g)^*$, $(u_0)^*$ $\underline{and}$ $(u_1)^*$ $\underline{be\ satisfied.}$ $\underline{If\ the}$ $L^1([0,\infty); L^2(R))$ $\underline{norms\ of}$ g, g_t, $\underline{the}$ $L^2([0,\infty); L^2(R))$ $\underline{norms\ of}$ g_x, g_{tt}, g_{tx} $\underline{and\ the}$ $L^2(R)$ $\underline{norms}$ $\underline{of}$ u_{0x}, u_{0xx}, u_{0xxx}, u_1, u_{1x}, u_{1xx} $\underline{are\ sufficiently\ small,\ then\ there\ is\ a\ unique}$ $\underline{global\ solution}$ $u(t,x) \in C^2([0,\infty) \times [0,1])$ $\underline{of}$ (VE)* $\underline{and}$

(6.4) $\qquad\qquad u_t, u_x, u_{tt}, u_{tx}, u_{xx}, u_{ttt}, u_{ttx}, u_{txx}, u_{xxx} \in L^\infty([0,\infty); L^2(0,1))$,

(6.5) $\qquad\qquad u_{tt}, u_{tx}, u_{xx}, u_{ttt}, u_{ttx}, u_{txx}, u_{xxx} \in L^2([0,\infty); L^2(0,1))$,

(6.6) $\qquad u_t(t,x), u_x(t,x), u_{tt}(t,x), u_{tx}(t,x), u_{xx}(t,x) \to 0, \; t \to \infty, \; \underline{unif.\ on} \; (0,1)$.

The proofs of Theorems 6.1 and 6.2 are identical to the proofs of Theorems 4.1 and 5.1, respectively. Indeed, virtually all equations of Sections 3-5 are valid also for the Neumann boundary value problem, provided that integration with respect to x over $(-\infty,\infty)$ be replaced by integration over $(0,1)$. In particular the crucial estimates (3.6)-(3.10), (4.9), (4.12), (4.13), (4.21), (4.22), (5.5), (5.8), (5.9), (5.17), (5.18) are valid. The reason is that the assumed Neumann boundary conditions annihilate the boundary contribution when, in the derivation of the estimates, we integrate by parts with respect to x.

The same observation holds if we replace the Neumann by Dirichlet boundary conditions,

(6.7) $\qquad\qquad\qquad u(t,0) = u(t,1) = 0, \qquad 0 < t < \infty$,

provided we impose on the forcing terms the condition

(6.8) $\qquad\qquad\qquad f(t,0) = f(t,1) = 0, \qquad 0 < t < \infty$,

(6.9) $\qquad\qquad\qquad g(t,0) = g(t,1) = 0, \qquad 0 < t < \infty$.

The reason here is that (6.7), (6.8), (6.9) and the equations in (HF)*, (VE)* imply

(6.10) $\qquad\qquad \sigma(u_x)_x(t,0) = \sigma(u_x)_x(t,1) = 0, \qquad 0 < t < \infty$,

so that boundary contributions are again annihilated when, in the derivation of the estimates, we integrate by parts with respect to x. We also note that for the Dirichlet boundary-initial value versions of (HF) and (VE) we can also conclude that $u(t,x) \to 0$, uniformly for $x \in [0,1]$, as $t \to \infty$. If one imposes boundary conditions (6.7) but without (6.8) and (6.9) the problem can be solved but the estimates need certain modifications that we shall not discuss here.

Finally, we remark that MacCamy [7], [8] studies the initial-boundary value problems for (HF) and (VE) only for the case of Dirichlet boundary conditions, and that his assumptions on the boundary values of the data, see assumptions (u_0), (f_3) of [7] and (U_2), (F_4) of [8], are different and more restrictive than ours.

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7. **Two-Dimensional Heat Flow with Memory**. In this section we outline the applicability
of the energy method to the initial value problem (compare with (HF)):

$$(7.1) \quad \begin{cases} u_t(t,x_1,x_2) + \int_0^t a(t - \tau)Au(\tau,x_1,x_2)d\tau = f(t,x_1,x_2) \\ \qquad\qquad\qquad\qquad\qquad\qquad (0 < t < \infty, \ x = (x_1,x_2) \ \epsilon \ \mathbb{R}^2) , \\ u(0,x_1,x_2) = u_0(x_1,x_2) \qquad\qquad (x \ \epsilon \ \mathbb{R}^2) , \\ Au = -\left\{ \dfrac{\partial}{\partial x_1}\left[\varphi\left(\sqrt{u_{x_1}^2 + u_{x_2}^2}\right)u_{x_1}\right] + \dfrac{\partial}{\partial x_2}\left[\varphi\left(\sqrt{u_{x_1}^2 + u_{x_2}^2}\right)u_{x_2}\right]\right\} . \end{cases}$$

We assume that the function $\varphi : \mathbb{R}^+ \to \mathbb{R}$ satisfies

(φ) $\qquad\qquad\qquad\qquad\qquad\qquad \varphi \ \epsilon \ C^4(\mathbb{R}^+)$ and $\varphi(0) > 0$.

The problem (7.1) represents a mathematical model for heat flow in an unbounded two-
dimensional body of material with memory.

Proceeding as in Section 2 we differentiate (7.1) with respect to t, we define
the resolvent kernel k of a' by equation (k) (we assume for the moment that a
satisfies (a_H), and that f is smooth), and we apply the procedure of Section 2 to
arrive at the following equivalent form of (7.1) (compare with (1.1))

$$(7.2) \quad \begin{cases} u_{tt}(t,x) + \dfrac{\partial}{\partial t}\int_0^t k(t - \tau)u_t(\tau,x)d\tau = -Au(t,x) + \Phi(t,x), \quad 0 < t < \infty, \quad x \ \epsilon \ \mathbb{R}^2 \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x) = f(0,x), \quad x \ \epsilon \ \mathbb{R}^2 \\ \Phi(t,x) = f_t(t,x) + k(0)f(t,x) + \int_0^t k'(t - \tau)f(\tau,x)d\tau, \quad 0 \leq t < \infty, \quad x \ \epsilon \ \mathbb{R}^2 . \end{cases}$$

We shall use (7.2) to obtain global results for (7.1) in a manner analogous to the
way (1.1) is used in the proof of Theorem 4.1. Since (7.1) and (7.2) have two space
dimensions, the energy method requires that we obtain estimates of various partial
derivatives of u up to order four in $L^\infty([0,\infty); L^2(\mathbb{R}^2))$ and in $L^2([0,\infty); L^2(\mathbb{R}^2))$
(rather than up to order three for (HF) in one space dimension). This means that for
technical reasons the kernel k and the function Φ in (7.2) have to be correspondingly
smoother, in order to permit the additional differentiations of (7.2). To be precise
we replace assumptions (a_H) regarding the kernel a in (7.1) by (compare with Section 2):

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$$(\tilde{a}_H) \quad \begin{cases} \text{(i)} \quad a \in C^4[0,\infty), \ a^{(j)}(t) \ \text{are bounded on} \ [0,\infty) \ \text{for} \ j = 0,1,2,3,4 \\[6pt] \text{(ii)} \quad a(0) = 1, \ a'(0) < 0 \\[6pt] \text{(iii)} \quad t^j a^{(m)}(t) \in L^1(0,\infty), \quad j = 0,1,2,3,4, \quad m = 0,1,2,3,4 \\[6pt] \text{(iv)} \quad \text{Re} \ \hat{a}(i\eta) > 0, \quad \eta \in \mathbb{R}. \end{cases}$$

One then easily has the following strengthening of Lemma 2.1.

Lemma 7.1. Let the assumptions $(\tilde{a}_H)$ be satisfied and let $k(t)$ be the resolvent kernel of $a'(t)$. Then

(i) $k(t) \in C^3[0,\infty)$; $k(t)$, $k'(t)$, $k''(t)$, $k'''(t)$ are bounded on $[0,\infty)$.

(ii) $k(t) = k_\infty + K(t)$; $k_\infty = \dfrac{1}{\hat{a}(0)}$; $K^{(m)}(t) \in L^1(0,\infty)$, $m = 0,1,2,3$.

(iii) The inequality $(2.6)_H$ holds for some $\alpha > 0$ and for every $v \in L^2(0,T)$.

With the aid of Lemma 7.1 (i), (ii) one readily verifies that if f in (7.1) satisfies the assumptions

$$(\tilde{f}) \quad f, f_t, f_{x_j}, f_{tx_j}, f_{x_i x_j}, f_{ttt}, f_{tx_i x_j}, f_{ttx_j}, f_{tttt}, f_{tttx_j}, f_{ttx_i x_j}, f_{tx_i x_j x_\ell} \in L^2([0,\infty); L^2(\mathbb{R})),$$
$$i,j,\ell = 1,2 \ ,$$

then the function Φ in (7.2) has the property

$$(\tilde{\Phi}) \quad \Phi, \Phi_t, \Phi_{x_j}, \Phi_{tt}, \Phi_{tx_j}, \Phi_{x_i x_j}, \Phi_{ttt}, \Phi_{ttx_i}, \Phi_{tx_i x_j} \in L^2([0,\infty); \mathbb{R}^2), \quad i,j = 1,2$$

(compare with (f), (Φ) in Section 1).

Concerning the initial datum u_0 we assume

$$(\tilde{u}_0) \quad u_{0x_j}, u_{0x_i x_j}, u_{0x_i x_j x_\ell}, u_{0x_i x_j x_\ell x_m} \in L^2(\mathbb{R}^2), \quad i,j,\ell,m = 1,2 \ .$$

Our global result for (7.1), analogous to Theorem 4.1 for (HF) is:

Theorem 7.1. Let the assumptions (φ), $(\tilde{a}_H)$, $(\tilde{f})$, $(\tilde{u}_0)$ be satisfied. If the $L^2([0,\infty); L^2(\mathbb{R}))$ norms of f and its partial derivatives in $(\tilde{f})$ and the $L^2(\mathbb{R}^2)$ norms of the partial derivatives of u_0 in $(\tilde{u}_0)$ are sufficiently small, then there is a unique global solution $u(t,x) \in C^2([0,\infty) \times \mathbb{R}^2)$ of (7.1) and

$$(7.3) \quad \begin{cases} u_t, u_{x_i}, u_{tt}, u_{tx_j}, u_{x_i x_j}, u_{ttt}, u_{ttx_j}, u_{tx_i x_j}, u_{x_i x_j x_\ell}, \\[6pt] u_{tttt}, u_{tttx_j}, u_{ttx_i x_j}, u_{tx_i x_j x_\ell}, u_{x_i x_j x_\ell x_m} \in L^\infty([0,\infty); L^2(\mathbb{R}^2)), \ i,j,\ell,m = 1,2 \ , \end{cases}$$

$$(7.4) \quad \begin{cases} u_t, u_{tt}, u_{tx_j}, u_{x_ix_j}, u_{ttt}, u_{ttx_j}, u_{tx_ix_j}, u_{x_ix_jx_\ell}, u_{tttt}, u_{tttx_j}, \\[2mm] u_{ttx_ix_j}, u_{tx_ix_jx_\ell}, u_{x_ix_jx_\ell x_m} \in L^2([0,\infty); L^2(\mathbb{R}^2)), \quad i,j,\ell,m = 1,2 \ , \end{cases}$$

$$(7.5) \quad \begin{cases} u_t(t,\cdot), u_{tt}(t,\cdot), u_{tx_i}(t,\cdot), u_{x_ix_j}(t,\cdot), u_{ttt}(t,\cdot), u_{ttx_i}(t,\cdot), \\[2mm] u_{tx_ix_j}(t,\cdot), u_{x_ix_jx_\ell}(t,\cdot) \to 0, \ \underline{\text{as}} \ t \to \infty, \ \underline{\text{in}} \ L^2(\mathbb{R}^2), \quad i,j,\ell = 1,2 \ , \end{cases}$$

$$(7.6) \quad \begin{cases} u_t(t,x), u_{x_j}(t,x), u_{tt}(t,x), u_{tx_j}(t,x), u_{x_ix_j}(t,x) \to 0, \ \underline{\text{as}} \ t \to \infty, \\[2mm] \underline{\text{uniformly for}} \ x \in \mathbb{R}^2, \quad i,j = 1,2 \ . \end{cases}$$

The proof of Theorem 7.1 is completely analogous to that of Theorem 4.1, although more complicated technically, and we confine our discussion to a few remarks. The existence of a unique local solution of the equivalent problem (7.2) is handled as in Section 3 by a fixed point argument. In analogy with Section 3, eq. $(\sigma)^*$, we replace temporarily (φ) by the stronger assumption

$$(\varphi)^* \qquad \varphi \in C^4(\mathbb{R}^+), \ \varphi(w) \geq p_0, \ w\varphi'(w) + \varphi(w) \geq p_0, \ p_0 > 0, \ w \in \mathbb{R}^+ \ ,$$

which guarantees that the operator A is uniformly elliptic. In place of the set of functions $X(M,T)$ of Section 3 we now define for any positive M and T the set $\tilde{X}(M,T)$ of functions $u(t,x) \in C^2([0,T] \times \mathbb{R}^2)$ with initial values $u(0,x) = u_0(x)$, $u_t(0,x) = u_1(x) = f(0,x)$, $x \in \mathbb{R}^2$, such that the partial derivatives in (7.3) are in $L^\infty([0,T]; L^2(\mathbb{R}^2))$ and which satisfy (compare with (3.3) for $n = 1$)

$$(7.7) \quad \sup_{[0,T]} \int_{-\infty}^\infty \int_{-\infty}^\infty \Bigg\{ u_t^2(t,x) + u_{tt}^2(t,x) + u_{ttt}^2(t,x) + u_{tttt}^2(t,x) + \sum_{i=1}^2 \Big(u_{x_i}^2(t,x) $$
$$+ u_{tx_i}^2(t,x) + u_{ttx_i}^2(t,x) + u_{tttx_i}^2(t,x) \Big) + \sum_{i,j=1}^2 \Big(u_{x_ix_j}^2(t,x) + u_{tx_ix_j}^2(t,x) $$
$$+ u_{ttx_ix_j}^2(t,x) \Big) + \sum_{i,j,\ell=1}^2 \Big(u_{x_ix_jx_\ell}^2(t,x) + u_{tx_ix_jx_\ell}^2(t,x) \Big) $$
$$+ \sum_{i,j,\ell,m=1}^2 u_{x_ix_jx_\ell x_m}^2(t,x) \Bigg\} dx_1 dx_2 \leq M^2 \ .$$

Note that $\tilde{X}(M,T)$ is nonempty for M sufficiently large. Observe also that if $u \in \tilde{X}(M,T)$, then by Sobolev's inequality (see e.g. [1, p. 32])

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$$(7.8) \qquad \sup_{[0,T]\times \mathbb{R}^2} \{|u_t(t,x)|, |u_{x_i}(t,x)|, |u_{tt}(t,x)|, |u_{tx_i}(t,x)|,$$

$$|u_{x_i x_j}(t,x)|\} \leq M \qquad (i,j = 1,2) ,$$

(compare with (3.4)). The fixed point argument, quite analogous to that of Theorem 3.1, is now applied to the map $\tilde{S} : \tilde{X}(M,T) \to C^2([0,T] \times \mathbb{R}^2)$ which carries a function $v \in \tilde{X}(M,T)$ into the solution $u(t,x)$ of the linear hyperbolic initial value problem (compare with (3.5))

$$(7.9) \qquad \begin{cases} u_{tt}(t,x) + k(0)u_t(t,x) = \varphi\left(\sqrt{v_{x_1}^2 + v_{x_2}^2}\right)\left[u_{x_1 x_1} + u_{x_2 x_2}\right] \\ \quad + \dfrac{\varphi'\left(\sqrt{v_{x_1}^2 + v_{x_2}^2}\right)}{\sqrt{v_{x_1}^2 + v_{x_2}^2}}\left[v_{x_1}^2 u_{x_1 x_1} + 2v_{x_1}v_{x_2}u_{x_1 x_2} + v_{x_2}^2 u_{x_2 x_2}\right] \\ \quad + \Phi(t,x) - (k'*v_t)(t,x) \qquad (0 \leq t < \infty, \ x \in \mathbb{R}^2) \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x) = f(0,x) \qquad (x \in \mathbb{R}^2) . \end{cases}$$

One arrives easily at analogues of Theorems 3.1, 3.2 for the existence, uniqueness, and regularity of the (generally) local solution $u(t,x) \in C^2([0,T_0) \times \mathbb{R}^2)$ of the Cauchy problem (7.2) on a maximal interval $[0,T_0)$, $T_0 \leq \infty$, such that for $T \in [0,T_0)$

$$(7.10) \qquad \begin{cases} u_t, u_{x_i}, u_{tt}, u_{tx_i}, u_{x_i x_j}, u_{ttt}, u_{ttx_i}, u_{tx_i x_j}, u_{x_i, x_j, x_\ell}, u_{tttt}, u_{tttx_i}, \\ u_{ttx_i x_j}, u_{tx_i x_j x_\ell}, u_{x_i x_j x_\ell x_m} \in L^\infty([0,T]; L^2(\mathbb{R}^2)) , \end{cases}$$

for $i,j,\ell,m = 1,2$ (compare with 3.2); moreover, if $T_0 < \infty$, then the integral in (7.7) $\to \infty$ as $t \to T_0^-$. We omit the details.

To complete the proof of Theorem 7.1 one proceeds to obtain energy estimates for derivatives of u as in Section 4 (however, to obtain the $L^\infty([0,\infty); L^2(\mathbb{R}^2)$ and the $L^2([0,\infty); L^2(\mathbb{R}^2))$ estimates of the fourth derivatives of u in (7.3), and (7.4), it is now necessary to differentiate (7.2) up to three times, as opposed to up to twice).

Similar to the proof of Theorem 4.1 we have to restrict the range of $u_{x_1}(t,x)$ and $u_{x_2}(t,x)$ to the set on which A is elliptic in order that (7.2) be a well posed problem. We choose a constant $c_0 > 0$ such that

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(7.11) $\varphi(w) \geq p_0$, $w\varphi'(w) + \varphi(w) \geq p_0$, $p_0 > 0$, $w \in [0, c_0]$.

We define $W(w) = \int_0^w \xi\varphi(\xi)d\xi$ and we note that

(7.12) $$W(w) \geq \frac{1}{2} p_0 w^2, \qquad w \in [0, c_0) .$$

As in Section 4 the aim is to show that there exists a number μ, $\mu < c_0$, such that if the local solution $u(t,x)$ of (7.2) satisfies (compare with (4.8))

(7.13) $$\sup_{[0,T]\times \mathbb{R}^2} \{|u_{x_i}(t,x)|, |u_{tx_i}(t,x)|, |u_{x_i x_j}(t,x)|\} \leq \mu, \quad (i,j = 1,2) ,$$

then certain functionals of the solution u are controllably small.

To obtain the first set of estimates we multiply (7.2) by u_t and integrate each term over $[0,s] \times \mathbb{R}^2$, $0 < s < T < T_0$. We make use of $(2.6)_H$ in Lemma 7.1 (iii), an integration by parts (compare with (4.9)), and we obtain

(7.14) $$\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_t^2(s,x)dx_1 dx_2 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W\left(\sqrt{u_{x_1}^2(s,x) + u_{x_2}^2(s,x)}\right)dx_1 dx_2$$

$$+ \alpha \int_0^s \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_t^2(t,x)dx_1 dx_2 dt \leq \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_t^2(0,x)dx_1 dx_2$$

$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W\left(\sqrt{u_{x_1}^2(0,x) + u_{x_2}^2(0,x)}\right)dx_1 dx_2 + \int_0^s \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(t,x)u_t(t,x)dx_1 dx_2 dt .$$

By an argument similar to that following (4.9) one obtains that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_t^2(s,x)dx_1 dx_2, \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_{x_i}^2(s,x)dx_1 dx_2 \quad (i = 1,2), \quad \int_0^s \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_t^2(t,x)dx_1 dx_2 dt$$

are controllably small, uniformly on $[0,T]$.

We omit the derivation of the remaining, technically involved, energy estimates which follow the pattern of those in Section 4. We only remark that for most of these calculations it is convenient to write Au in the form

$$Au = -\varphi\left(\sqrt{u_{x_1}^2 + u_{x_2}^2}\right)\{u_{x_1 x_1} + u_{x_2 x_2}\} - \frac{\varphi'\left(\sqrt{u_{x_1}^2 + u_{x_2}^2}\right)}{\sqrt{u_{x_1}^2 + u_{x_2}^2}}\{u_{x_1}^2 u_{x_1 x_1} + 2u_{x_1}u_{x_2}u_{x_1 x_2} + u_{x_2}^2 u_{x_2 x_2}\} .$$

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Once we have obtained the $L^\infty([0,T]; L^2(\mathbb{R}^2))$ and $L^2([0,T]; L^2(\mathbb{R}^2))$ estimates of derivatives of u up to order four, we can select initial data and forcing term so "small" that, for as long as (7.13) is satisfied, one has (compare with (4.30))

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{u_t^2 + u_{tt}^2 + u_{ttt}^2 + u_{tttt}^2 + \sum_{i=1}^{2} (u_{x_i}^2 + u_{tx_i}^2 + u_{ttx_i}^2 + u_{tttx_i}^2)$$

$$+ \sum_{i,j=1}^{2} (u_{x_ix_j}^2 + u_{tx_ix_j}^2 + u_{ttx_ix_j}^2) + \sum_{i,j,\ell=1}^{2} (u_{x_ix_jx_\ell}^2 + u_{tx_ix_jx_\ell}^2)$$

$$+ \sum_{i,j,\ell,m=1}^{2} u_{x_ix_jx_\ell x_m}^2 \} dx_1 dx_2 + \int_0^s \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{u_t^2 + u_{tt}^2 + u_{ttt}^2 + u_{tttt}^2$$

$$+ \sum_{i=1}^{2} (u_{tx_i}^2 + u_{ttx_i}^2 + u_{tttx_i}^2) + \sum_{i,j=1}^{2} (u_{x_ix_j}^2 + u_{tx_ix_j}^2 + u_{ttx_ix_j}^2)$$

$$+ \sum_{i,j,\ell=1}^{2} (u_{x_ix_jx_\ell}^2 + u_{tx_ix_jx_\ell}^2) + \sum_{i,j,\ell,m=1}^{2} u_{x_ix_jx_\ell x_m}^2 \} dx_1 dx_2 dt \le \mu^2 ,$$

and one concludes the proof of Theorem 7.1 exactly as that of Theorem 4.1.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We use energy methods to study global existence, boundedness, and asymptotic behavior as $t \to \infty$, of solutions of the two Cauchy problems (and related initial-boundary value problems)
approaches infinity

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20. ABSTRACT (cont'd.)

$$
\text{(HF)}\quad
\begin{cases}
u_t(t,x) = \displaystyle\int_0^t a(t-\tau)\sigma(u_x(\tau,x))_x\,d\tau + f(t,x) & (0 < t < \infty,\ x \in \mathbb{R})\\[2mm]
u(0,x) = u_0(x) & (x \in \mathbb{R}),
\end{cases}
$$

$$
\text{(VE)}\quad
\begin{cases}
u_{tt}(t,x) = \sigma(u_x(t,x))_x + \displaystyle\int_0^t a'(t-\tau)\sigma(u_x(\tau,x))_x\,d\tau + g(t,x)\\[2mm]
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (0 < t < \infty,\ x \in \mathbb{R})\\[2mm]
u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) & (x \in \mathbb{R})
\end{cases}
$$

with suitably "small" data u_0, u_1, f, g; (HF) and (VE) are mathematical models
for nonlinear one-dimensional heat flow in a material with "memory" and non-
linear one-dimensional viscoelastic motion, respectively. Here $a : [0,\infty) \to \mathbb{R}$,
$\sigma : \mathbb{R} \to \mathbb{R}$, $f,g : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, $u_0, u_1 : \mathbb{R} \to \mathbb{R}$ are given, sufficiently
smooth functions; the subscripts x or t denote partial derivatives. If
$a(0) = 1$ formal differentiation with respect to t reduces (HF) to (VE) with
$g(t,x) = f_t(t,x)$ and $u_1(x) = f(0,x)$. But, since (HF) and (VE) have different
physical origins, the corresponding natural assumptions concerning $a(\cdot)$ are
drastically different and, therefore, the two problems are studied separately.

A previous study of (HF) and (VE) rests on the concept of Riemann invariant
and is restricted to one space dimension. The energy method is simpler in
principle and yields more widely applicable results.